CS261: Exercise Set #5

For the week of April 27–May 1, 2015

Instructions:

(1) *Do not turn anything in.*

(2) The course staff is happy to discuss the solutions of these exercises with you in office hours or on Piazza.

(3) While these exercises are certainly not trivial, you should be able to complete them on your own (perhaps after consulting with the course staff or a friend for hints).

Exercise 21

In the *multicommodity flow problem*, the input is a directed graph $G = (V, E)$ with k source vertices $s_1, \ldots, s_k$, k sink vertices $t_1, \ldots, t_k$, and a nonnegative capacity u_e for each edge $e \in E$. An s_i-t_i pair is called a *commodity*. A *multicommodity flow* if a set of k flows $\mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(k)}$ such that (i) for each $i = 1, 2, \ldots, k$, $\mathbf{f}^{(i)}$ is an s_i-t_i flow (in the usual max flow sense); and (ii) for every edge e, the total amount of flow (summing over all commodities) sent on e is at most the edge capacity u_e. The *value* of a multicommodity flow is the sum of the values (in the usual max flow sense) of the flows $\mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(k)}$.

Prove that the problem of finding a multicommodity flow of maximum-possible value reduces in polynomial time to solving a linear program.

Exercise 22

In the *set cover problem*, the input is a universe U of elements and a list $S_1, \ldots, S_m \subseteq U$ of subsets. In addition, each set S_i has a nonnegative weight w_i. The goal is pick the collection of S_i's that covers all of the elements of U and, subject to this, has the minimum total weight. (Assume that each element belongs to at least one of the sets.) This problem is equivalent to the following integer program:

$$\min \sum_{i=1}^{m} w_i x_i$$

subject to

$$\sum_{i\,:\,e \in S_i} x_i \geq 1 \qquad \text{for every } e \in U$$

$$x_i \in \{0, 1\} \qquad \text{for every } i = 1, 2, \ldots, m.$$

Consider the linear programming relaxation of this integer program, where the constraints that $x_i \in \{0, 1\}$ for each i are replaced by the linear constraint that $x_i \geq 0$ for each i. What is the dual of this linear program?

Exercise 23

In the *(unweighted) independent set* problem, the input is an undirected graph $G = (V, E)$. The goal is to compute a maximum-cardinality subset of vertices $S \subseteq V$ such that no edge has both of its endpoints in S. Show that this problem can be expressed as an integer program, with one 0-1 variable per vertex and one linear constraint per edge. Show by explicit example that the maximum objective function value of the linear programming relaxation (with the 0-1 constraints replaced by nonnegativity constraints) can be strictly larger than the maximum size of an independent set of G.[1]

Exercise 24

The minimum spanning tree problem (for an undirected graph $G = (V, E)$ with an edge cost c_e per edge e) can be formulated as the following integer program (why?):

$$\min \sum_{e \in E} c_e x_e$$

subject to

$$\sum_{e \in \delta(S)} x_e \geq 1 \qquad \text{for every non-empty strict subset } S \text{ of } V^2$$

$$x_e \in \{0, 1\} \qquad \text{for every } e \in E.$$

Consider the linear programming relaxation of this integer program, where the 0-1 constraints are replaced by the constraints $x_e \geq 0$ for every $e \in E$. Can the optimal value of this linear program be strictly less than that of an MST of G? Either show by example that it can, or provide a proof that it can't.

Exercise 25

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$ be a set of n m-vectors. Define C as the *cone* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, meaning all linear combinations of the $\mathbf{x}_i$'s that use only nonnegative coefficients:

$$C = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{x}_i \; : \; \lambda_1, \ldots, \lambda_n \geq 0 \right\}.$$

Suppose $\alpha \in \mathbb{R}^m$, $\beta \in \mathbb{R}$ define a valid inequality for C, meaning that

$$\alpha^T \mathbf{x} \geq \beta$$

for every $\mathbf{x} \in C$. Prove that

$$\alpha^T \mathbf{x} \geq 0$$

for every $\mathbf{x} \in C$, so α and 0 also define a valid inequality.

[Hint: Use the fact that if $\mathbf{x} \in C$ then $\lambda \mathbf{x} \in C$ for all scalars $\lambda \geq 0$.]

[1] This is to be expected, as the problem is NP-hard.
[2] Recall that $\delta(S)$ denotes the edges with exactly one endpoint in S.