COMS 4995 (Randomized Algorithms): Exercise Set #1

For the week of September 2–6, 2019

Instructions:

(1) *Do not turn anything in.*

(2) The course staff is happy to discuss the solutions of these exercises with you in office hours or in the course discussion forum.

(3) While these exercises are certainly not trivial, you should be able to complete them on your own (perhaps after consulting with the course staff or a friend for hints).

Exercise 0

Watch the two "probability review" videos posted on the course Web site (`cs.columbia.edu/~tr/f19/f19.html`); or, if you remember the concepts of state spaces, events, random variables, expectation, conditional probabilities and expectations, and independence (of events and of random variables), feel free to skip.

Exercise 1

Suppose we need an algorithm that assigns processes to servers, but we're feeling super-lazy. One easy solution is to just assign each process to a random server, with each server equally likely. How well does this work?

For concreteness, assume there are n processes and also n servers, where n is some positive integer. The sample space is all n^n possible ways of assigning the processes to the servers, with n choices for each of the n processes. By the definition of our lazy algorithm, each of these n^n outcomes is equally likely.

Now that we have a sample space, we're in a position to define random variables. One interesting quantity is server load, so let's define Y as the random variable equal to the number of processes that get assigned to the first server. (The story is the same for all the servers by symmetry, so we may as well focus on the first one.) What is the expectation of Y? Formally prove your answer.

Exercise 2

Consider a group of k people. Assume that each person's birthday is drawn uniformly at random from the 365 possibilities. (And ignore leap years.) What is the smallest value of k such that the expected number of pairs of distinct people with the same birthday is at least one? Formally prove your answer. [Hint: Define an indicator random variable for each pair of people. Use linearity of expectation.]

Exercise 3

Prove *Markov's inequality:* if X is a non-negative random variable with finite expectation and $c > 1$, then

$$\mathbf{Pr}[X \geq c \cdot \mathbf{E}[X]] \leq \frac{1}{c}.$$