COMS 4995 (Randomized Algorithms): Exercise Set #5

For the week of September 30–October 4, 2019

Instructions:

(1) *Do not turn anything in.*

(2) The course staff is happy to discuss the solutions of these exercises with you in office hours or in the course discussion forum.

(3) While these exercises are certainly not trivial, you should be able to complete them on your own (perhaps after consulting with the course staff or a friend for hints).

Exercise 21

Show that Chebyshev's inequality is nearly tight, in the following sense: for arbitrarily large positive integers t, there is a random variable X with the following properties:

1. $\mathbf{E}[X] = O(1)$ (e.g., at most 2).
2. $\mathrm{Var}[X] = O(1)$ (e.g., at most 4).
3. $\mathbf{Pr}[|X - \mathbf{E}[X]\,| \geq t \cdot \mathrm{StdDev}[X]] = \Omega\left(\frac{1}{t^2}\right)$, where the constant hidden in the big-omega is independent of t.

[Hint: Consider throwing n balls into n bins. But instead of doing it uniformly, randomize only over outcomes where one bin gets lots of balls and the other bins get zero or one ball each.]

Exercise 22

In Lecture #9 we proved that if X is a standard Gaussian (i.e., with mean 0 and variance 1), then for every $a \geq 0$,

$$\mathbf{Pr}[X \geq a] \leq e^{-a^2/2}.$$

Derive from this the following inequality, which massively improves over Chebyshev's inequality: for a Gaussian random variable with mean μ and variance σ^2/n,

$$\mathbf{Pr}[|X - \mathbf{E}[X]\,| > \epsilon] \leq 2e^{-n\epsilon^2/2\sigma^2}.$$

(We're using variance σ^2/n to match up with our application of Chebyshev's inequality to averages of n i.i.d. random variables each with variance σ^2.)

Exercise 23

In Lecture #9 we proved the following:

1. Scaling a standard Gaussian random variable by σ results in a Gaussian with mean 0 and variance σ^2. (Actually, this is by definition.)
2. Adding τ to a Gaussian random variable with mean μ and variance σ^2 yields a Gaussian with mean $\mu + \tau$ and variance σ^2. (Again, by definition.)

3. The sum of two independent standard Gaussian random variables is a Gaussian with mean 0 and variance 2. (This was the proof where we rotated the axes to make our double integral easy to evaluate.)

(a) Extend the third point above to a sum of two independent mean-0 Gaussians with arbitrary variances. (I.e., prove that if $X \sim \mathcal{N}(0, \sigma_1^2)$ and $Y \sim \mathcal{N}(0, \sigma_2^2)$, then $X + Y \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.)

[Hint: Use the same idea but modify the boundary of the integration region appropriately.]

(b) Extend the third point above to a sum of two independent Gaussians with arbitrary means and variances. (I.e., prove that if $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$, then $X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.)