

# Optimal Platform Design

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*An auction house cannot generally provide the optimal auction technology to every client. Instead it provides one or several auction technologies, and clients select the most appropriate one. For example, eBay provides ascending auctions and “buy-it-now” pricing. For each client the offered technology may not be optimal, but it would be too costly for clients to create their own. We call these mechanisms, which emphasize generality rather than optimality, platform mechanisms. A platform mechanism will be adopted by a client if its performance exceeds that of the client’s outside option, e.g., hiring (at a cost) a consultant to design the optimal mechanism. We ask two related questions. First, for what costs of the outside option will the platform be universally adopted? Second, what is the structure of good platform mechanisms? We answer these questions using a novel prior-free analysis framework in which we seek mechanisms that are approximately optimal for every prior.*

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Auction houses, like Sotheby's, Christie's, and eBay, exemplify the commodification of economic mechanisms, like auctions, and warrant an accompanying theory of design. The field of *mechanism design* suggests how special-purpose mechanisms might be optimally designed; however, in commodity industries there is a trade-off between special-purpose and general-purpose products. While for any particular setting an optimal special-purpose product is better, a general-purpose product may be favored, for instance, because of its cheaper cost or greater versatility. We develop a theory for the design of general-purpose mechanisms, henceforth, *platform design*.

Consider the following simple model for platform design. The platform *provider* offers a *platform* mechanism to potential customers (*principals*), who each wish to employ the mechanism in their particular *setting*. For example, the provider is eBay, the platform is the eBay auction, the principals are sellers, and the settings are the distinct markets of the sellers, which comprise of a set of buyers (*agents*) with preferences drawn according to a distribution. Each principal has the option to not adopt the platform and instead to employ a consultant to design the optimal auction for his specific setting. We assume that this outside option comes at a greater cost than the platform, and thus the platform provider has a *competitive advantage*.

We impose two restrictions to focus on the differences between the special-purpose optimal mechanism design and the general-purpose optimal platform design. First, we restrict the platform to be a single, unparameterized mechanism (unlike eBay where sellers can set their own reserve prices).<sup>1</sup> Second, we require that the platform is universally adopted. Without this

<sup>1</sup>In a separate study, we consider the technically orthogonal topic of reserve-price based platforms (Hartline and Roughgarden, 2009).

assumption, we would need to model in detail the relative value of adoption in each setting, and this would likely give less general results. We ask: *What must the competitive advantage of the platform be to guarantee universal adoption by all principals? What is the platform designer's mechanism that guarantees universal adoption?*

For several important objectives for mechanism design we show that there are platform mechanisms that are universally adopted with modest competitive advantage. Moreover these platform mechanisms are fundamentally different from the standard mechanisms that arise in special-purpose mechanism design. For example, standard mechanisms operate on an absolute scale, while platform mechanisms operate on a relative scale. What is important for the platform is not whether an agent has a high or low value, but whether an agent has a high or low value relative to the values of the other agents. The platform design question exposes the challenge of determining, while the mechanism is being run, the relevant scale. This distinction is critical; for potentially large markets no standard mechanism is universally adopted with finite competitive advantage.

There are two important points of contact between this theory of platform design and the existing literature. First, the problem of optimal platform design provides a formal setting in which to explore the Wilson (1987) doctrine, which critiques mechanisms that are overly dependent on the details of the setting but does not quantify the cost of this dependence. A universally adopted platform, by definition, performs well in all settings and hence is not dependent on the details of setting. Second, the optimal platform design problem is closely related to *prior-free optimal mechanism design*. Indeed, our study of platform design formally connects the prior-free

and Bayesian theories of optimal mechanism design. We make a rigorous comparison between the two settings and quantify the Bayesian designer's relative advantage over the prior-free designer.

#### PLATFORM DESIGN.

In classical Bayesian optimal mechanism design, a principal designs a mechanism for a set of self-interested agents that have private preferences over the outcomes of the mechanism. These private preferences are drawn from a known probability distribution. The optimal mechanism is the one that maximizes the expected value of the principal's objective function when the agents' strategies are in Bayes-Nash equilibrium.

For a given distribution and objective function, the *approximation factor* of a candidate mechanism is the ratio between the expected performance of an optimal mechanism and that of the candidate mechanism. A good mechanism is one with a small approximation factor (close to 1); a bad one has a large approximation factor.

We assume that the cost of designing the optimal mechanism is higher than the cost of adopting the platform. For this reason, a principal might choose to adopt the sub-optimal platform mechanism. We assume this *competitive advantage* of the platform is multiplicative. This assumption is consistent with commission structures in marketing and, from a technical point of view, frees the model from artifacts of scale. The platform's competitive advantage gives an upper bound on the approximation factor that the platform mechanism needs to induce a principal to adopt the platform instead of hiring a consultant to design the optimal mechanism. Each principal's decision to adopt is based on the platform mechanism's performance in the

principal's setting. Therefore, universal adoption demands that the platform mechanism's approximation factor on every distribution is at most its competitive advantage. Of particular interest is the minimum competitive advantage for which there is a platform that is universally adopted, and also the platform that attains this minimum approximation factor. This *optimal platform* is the mechanism that minimizes (over mechanisms) the maximum (over distributions) approximation factor. Optimal platform design is therefore inherently a min-max design criterion.

The basic formal question of platform design is: *What is the minimum competitive advantage  $\beta$  and optimal platform mechanism  $\mathcal{M}$  such that for all distributions  $F$  the expected performance of  $\mathcal{M}$  when values are drawn i.i.d. from  $F$  is at least  $\frac{1}{\beta}$  times the expected performance of the optimal mechanism for  $F$ ?*

Directly answering the platform design questions above is difficult as it requires simultaneous consideration of all distributions. This difficulty motivates a more stringent version of the basic question which has the following economic interpretation. Suppose that instead of requiring the principal to choose ex ante between the optimal mechanism and the platform, we allow him to choose ex post? Clearly, this makes the platform designer's task even more challenging, in that the minimum achievable  $\beta$  is only higher.

The formal question of platform design now becomes: *What is the minimum competitive advantage  $\beta$  and optimal platform mechanism  $\mathcal{M}$  such that for all valuation profiles  $\mathbf{v} = (v_1, \dots, v_n)$  the performance of  $\mathcal{M}$  on  $\mathbf{v}$  is at least  $\frac{1}{\beta}$  times the supremum over symmetric<sup>2</sup> Bayesian optimal mechanisms'*

<sup>2</sup>Our study focuses solely on settings where the agents are a priori indistinguishable. This focus motivates our restriction to i.i.d. distributions and symmetric optimal mechanisms. Distinguishable agents are considered by Balcan et al. (2008) and Bhattacharya

*performance on  $\mathbf{v}$ ?*

This question motivates the definition of a *performance benchmark* that is defined point-wise on valuation profiles, specifically as the supremum over optimal symmetric mechanisms' performance on the given valuation profile. Notice that this benchmark is prior-free. The analysis of a platform mechanism is then a comparison of the performance of a prior-free platform mechanism and a prior-free performance benchmark.

## RESULTS.

Our contributions are two-fold. First, we propose a conceptual framework for the design and analysis of general-purpose platforms. Second, we instantiate this framework to derive novel platform mechanisms for specific problems and, in some cases, prove their optimality.

In more detail, we consider the problem of optimal platform design in general symmetric settings of multi-unit unit-demand allocation problems and for general linear (in agents' payments and values) objectives of the principal. For much of the paper, we focus on the canonical objective of *consumer surplus*, which is the difference between the winning agents' values and payments. Consumer surplus is interesting in its own right (e.g., McAfee and McMillan, 1992; Condorelli, 2012; Chakravarty and Kaplan, 2013) and is, in a sense, technically more general than the objectives of surplus and profit.<sup>3</sup> Intuitively, maximizing the consumer surplus involves compromising between the competing goals of identifying high-valuation

et al. (2013).

<sup>3</sup>For surplus maximization, the Vickrey auction is optimal for every distribution. For profit maximization, reserve-price-based auctions are optimal under standard distributional assumptions (Myerson, 1981). For consumer surplus, reserve-price-based auctions are not optimal even for standard distributions.

agents and of minimizing payments. For example, with a single item, the Vickrey auction performs well when there is only one high-valuation agent; while the lottery, which gives the item away for free, is good when all agents have comparable valuations.

Our approach comprises four steps.

- 1) We characterize Bayesian optimal mechanisms for multi-unit unit-demand allocation problems and general linear objectives by a straightforward generalization of the literature on optimal mechanism design.
- 2) We characterize the prior-free performance benchmark, i.e., the supremum over optimal symmetric mechanisms' performance on a given valuation profile, as an ex post optimal two-level lottery.
- 3) We give a general platform design and a finite upper bound on the competitive advantage necessary for universal adoption.
- 4) We give a lower bound on the competitive advantage for which there exists a platform that achieves universal adoption.

Importantly, the platform mechanisms that we identify as being universally adopted with finite competitive advantage are not standard mechanisms from the literature on Bayesian optimal mechanisms. Indeed, we prove that no standard mechanism is universally adopted with any finite competitive advantage. Instead, general purpose mechanisms for platforms require novel features, which we identify in Step 3.

#### EXAMPLE.

Our main results are interesting to interpret in the special case of allocating a single item to one of two agents to maximize the consumer surplus.

Denote the high agent value by  $v_{(1)}$  and the low agent value by  $v_{(2)}$ . We characterize the performance benchmark as  $\max(\frac{v_{(1)}+v_{(2)}}{2}, v_{(1)} - \frac{v_{(2)}}{2})$ . As the supremum of Bayesian optimal mechanisms, the first term in this benchmark arises from a lottery and the second term from the two-level lottery that serves a random agent with value strictly above price  $v_{(2)}$  if one exists and otherwise serves a random agent (at price zero). The optimal platform mechanism randomizes between a lottery and weighted Vickrey auctions. Precisely, it sets  $w_1 = 1$ , draws  $w_2$  uniformly from  $\{0, 1/2, 2, \infty\}$ , and serves the agent  $i \in \{1, 2\}$  that maximizes  $w_i v_i$ . It is universally adopted with competitive advantage  $\frac{4}{3}$  and no other mechanism is better.

While all possible prior distributions are considered when deriving the performance benchmark above, the actual benchmark for a particular valuation profile is given by a simple formula with no distributional dependence. Consequently, our analysis that shows the  $\frac{4}{3}$  competitive advantage is a simple comparison between a (prior-free) platform mechanism and a (prior-free) performance benchmark in the worst case over valuation profiles.

#### RELATED WORK.

Our description of Bayesian optimal mechanisms for general linear objectives follows from the work on optimal mechanism design (see Myerson, 1981, and Riley and Samuelson, 1981). Within this theory, the consumer surplus objective coincides with that of the grand coalition in a *weak cartel*, where agents wish to maximize the cartel's total utility without side payments amongst themselves, so that payments to the auctioneer are effectively "burnt". Our characterizations are thus related to those in the literature on collusion in multi-unit auctions, e.g., by McAfee and McMil-

lan (1992) and Condorelli (2012). Recently, Chakravarty and Kaplan (2013) also specifically studied Bayesian optimal auctions for consumer surplus.

There is a growing literature on “redistribution mechanisms” where, similar to the objective of consumer surplus, payments are bad, e.g., see Moulin (2009) and Guo and Conitzer (2009). These mechanisms transfer some of the winners’ payments back to the losers so that the residual payment left over is as small as possible. The mechanisms considered are prior-free.

As already mentioned, there is a large related literature on prior-free optimal mechanism design. Goldberg et al. (2001), Segal (2003), Baliga and Vohra (2003), and Balcan et al. (2008) consider asymptotic approximation of the Bayesian optimal mechanisms by a single (prior-free) mechanism. A key aspect of these works is the order of the existential quantification on the agents’ value distribution and the limit argument on the number of agents. These papers show that for all distributions, in the limit of the number of agents, their mechanisms performs well. Thus, their results do not address the question of whether or not a principal in a small or moderate-sized market would adopt the platform. In contrast, the line of research initiated by Goldberg et al. (2006) on prior-free profit maximization can be reinterpreted in the context of platform design; Section V describes this connection in detail.

Finally, there is an important and growing literature on minmax analyses in areas related to mechanism design. Like our framework for platform design, these analyses look for mechanisms that work well in the worst case when some of the fundamentals of the setting are unknown to the principal. Frankel (2014) applies such an analysis to the principal-agent problem of delegation; Carroll (2015) applies such an analysis to contract

design; and Carroll (2016) applies such an analysis to multi-dimensional screening. The conclusion of these studies is that while optimal mechanisms given the parameters can be complex, the minmax optimal mechanism often takes a simple and natural form.

### I. Warm-up: Monopoly Pricing

Consider the following monopoly pricing problem. A monopolist seller (principal) of a single item faces a single buyer (agent). The seller has no value for the item and wishes to maximize his revenue, i.e., the payment of the buyer. The buyer's value for the item is  $v \in [1, h]$  and she wishes to maximize her utility which is her value less her payment. The seller may post a price  $p$  and the buyer may take it or leave it. The buyer will clearly take any price  $p \leq v$ .

The seller's optimal mechanism, when the buyer's value comes from the distribution  $F$  (where  $F(z) = \Pr[v \leq z]$ ), is to post the price  $p$  that maximizes  $p(1 - F(p))$ , a.k.a., the *monopoly price*. The performance benchmark  $\mathcal{G}(v)$ , i.e., the revenue of the best of the Bayesian optimal mechanism when the buyer's value is  $v$ , is then  $\mathcal{G}(v) = v$ . The platform designer must give a single mechanism with revenue that approximates  $v$  for every value  $v$  in the support  $[1, h]$ . The optimal platform and its competitive advantage for universal adoption are given by the theorem below.

**THEOREM 1:** *The optimal platform mechanism offers a price drawn from distribution  $P$  with cumulative distribution function  $P(z) = (1 + \ln z)/(1 + \ln h)$  on  $[1, h]$ , and a point mass of  $1/(1 + \ln h)$  at 1, and is universally adopted with competitive advantage  $1 + \ln h$ .*

**PROOF:** An easy calculation verifies that, for every  $v \in [1, h]$ , the ex-

pected revenue from such a random price from  $P$  is  $v/(1 + \ln h)$ . Thus, the competitive advantage for universal adoption is  $1 + \ln h$  as claimed.

To show that this is the optimal platform, we can similarly find a distribution  $F$  over values  $v$  such that the expected revenue of every platform mechanism is 1. The *equal revenue* distribution has distribution function  $F(z) = 1 - 1/z$ , a point mass of  $1/h$  at  $h$ , and any price  $p$  is accepted by the agent with probability  $1/p$  for an expected revenue of 1. The expected value of the benchmark for the equal-revenue distribution can be calculated as  $\mathbf{E}[\mathcal{G}(v)] = \mathbf{E}[v] = 1 + \ln h$ . Thus, the ratio of these expectations is  $1 + \ln h$ , and for any platform mechanism there must be some  $v \in [1, h]$  that achieves the ratio. We conclude that no platform is universally adopted with competitive advantage less than  $1 + \ln h$ . Q.E.D.

This analysis can be viewed as a zero-sum game between the platform designer and Nature where the solution is a mixed strategy on the part of both players, every action in the game achieves equal payoff, and the value of the game is the optimal competitive advantage.

To conclude, we considered a simple monopoly pricing setting and derived for it the optimal platform. While a logarithmic competitive advantage may seem impractical, except when the maximum variation  $h$  in values is small, the ideas from this design and analysis play an important role in the developments of this paper. The platform mechanisms we derive subsequently, however, will be universally adopted with a competitive advantage that is an absolute constant, independent of the number of agents, the number of units, and the range of agent values.

## II. Review of Bayesian Optimal Mechanism Design

In this section we review Bayesian optimal mechanism design for single-dimensional agents, i.e., with utility given by the value for receiving a good or service less the required payment, and develop the notation employed in the remainder of the paper. Characterizing Bayesian optimal mechanisms is the first step in our approach to platform design.

We consider mechanisms for allocating  $k$  units of an indivisible item to  $n$  unit-demand agents. The outcome of such a mechanism is an *allocation vector*,  $\mathbf{x} = (x_1, \dots, x_n)$ , where  $x_i$  is 1 if agent  $i$  receives a unit and 0 otherwise, and a non-negative *payment vector*,  $\mathbf{p} = (p_1, \dots, p_n)$ . The allocation vector  $\mathbf{x}$  is required to be feasible, i.e.,  $\sum_i x_i \leq k$ , and we denote this set of feasible allocation vectors by  $\mathcal{X}$ .

We assume that each agent  $i$  is risk-neutral, has a privately known valuation  $v_i$  for receiving a unit, and aims to maximize her (quasi-linear) utility, defined as  $u_i = v_i x_i - p_i$ . Each agent's value is drawn independently and identically from a continuous distribution  $F$ , where  $F(z)$  and  $f(z)$  denote the cumulative distribution and density functions, respectively. We denote the *valuation profile* by  $\mathbf{v} = (v_1, \dots, v_n)$ .

We consider general symmetric, linear objectives of the mechanism designer. For valuation coefficient  $\gamma_v$  and payment coefficient  $\gamma_p$ , the objective for maximization is:

$$(1) \quad \sum_{i=1}^n \gamma_v v_i x_i + \gamma_p p_i.$$

We single out three such objectives: *surplus* with  $\gamma_v = 1$  and  $\gamma_p = 0$ , *profit* with  $\gamma_v = 0$  and  $\gamma_p = 1$ , and *consumer surplus* with  $\gamma_v = 1$  and

$\gamma_p = -1$ . We will not discuss surplus maximization in this paper as the optimal mechanism for this objective is simply the prior-free  $k$ -unit Vickrey auction; therefore, we assume that  $\gamma_p \neq 0$ .

We assume that agents play in Bayes-Nash equilibrium and moreover if truth-telling is a Bayes-Nash equilibrium then agents truth-tell. When searching for Bayesian optimal mechanisms, the revelation principle (Myerson, 1981) allows us to restrict attention to Bayesian incentive compatible mechanisms, i.e., ones with a truth-telling Bayes-Nash equilibrium.

#### A. Characterization of incentive compatibility.

The *allocation rule*,  $\mathbf{x}(\mathbf{v})$ , is the mapping (in equilibrium) from agent valuations to the outcome of the mechanism. Similarly the *payment rule*,  $\mathbf{p}(\mathbf{v})$ , is the mapping from valuations to payments. Given an allocation rule  $\mathbf{x}(\mathbf{v})$ , let  $x_i(v_i)$  be the interim probability with which agent  $i$  is allocated when her valuation is  $v_i$  (over the probability distribution on the other agents' valuations):  $x_i(v_i) = \mathbf{E}_{\mathbf{v}_{-i}}[x_i(v_i, \mathbf{v}_{-i})]$ . Similarly define  $p_i(v_i)$ . We require interim individual rationality, i.e., that non-participation in the mechanism is an allowable agent strategy. The following lemma provides the standard characterization of allocation rules that are implementable by Bayesian incentive compatible mechanisms and the accompanying payment rule (which is unique up to additive shifts, and usually fixed by setting  $p_i(0) = 0$ ).

LEMMA 1: (Myerson, 1981) *Every Bayesian incentive compatible mechanism satisfies, for all  $i$  and  $v_i \geq v'_i$ :*

(a) *Allocation monotonicity:  $x_i(v_i) \geq x_i(v'_i)$ .*

(b) *Payment identity:  $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z) dz + p_i(0)$ .*

## VIRTUAL VALUATIONS.

Myerson (1981) defined *virtual valuations* and showed that the virtual surplus of an agent is equal to her expected payment. For  $v \sim F$ , this *virtual valuation for payment* is:

$$(2) \quad \varphi(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}.$$

LEMMA 2: (Myerson, 1981) *In a Bayesian incentive-compatible mechanism with allocation rule  $\mathbf{x}(\cdot)$ , the expected payment of an agent equals her expected virtual surplus:  $\mathbf{E}_{\mathbf{v}}[p_i(\mathbf{v})] = \mathbf{E}_{\mathbf{v}}[\varphi(v_i) x_i(\mathbf{v})]$ .*

The notion of virtual valuations applies generally to linear objectives. By substituting virtual values for payments into the objective (1) we arrive at a formula for *general virtual values*:  $\vartheta(v_i) = (\gamma_v + \gamma_p)v_i - \gamma_p \frac{1-F_i(v_i)}{f_i(v_i)}$ . For the objective of consumer surplus, i.e., the sum of the agent utilities, *virtual values for utility* are given by:

$$(3) \quad \vartheta(v_i) = \frac{1-F_i(v_i)}{f_i(v_i)}.$$

The revenue-optimal mechanism for a given distribution is the one that maximizes the virtual surplus for payment subject to feasibility and monotonicity of the allocation rule. Analogously, optimal mechanisms for general linear objectives are precisely those that maximize the expected (general) virtual surplus subject to feasibility and monotonicity of the allocation rule. Unfortunately, choosing  $\mathbf{x}$  to maximize  $\sum_i \vartheta(v_i)x_i$  for each valuation profile  $\mathbf{v}$  does not generally result in a monotone allocation rule. When  $\vartheta(\cdot)$  is not monotone increasing, an increase in an agent's value may decrease her vir-

tual value and cause her to be allocated less frequently. Notice that under the standard “monotone hazard rate” assumption the virtual value function for utility  $\vartheta(v) = \frac{1-F_i(v)}{f_i(v)}$  is monotone *in the wrong direction*.

### IRONING.

We next generalize the “ironing” procedure of Myerson (1981) that transforms a possibly non-monotone virtual valuation function into an *ironed virtual valuation* function that is monotone; optimizing ironed virtual surplus results in a monotone allocation rule. Furthermore, the ironing procedure preserves the target objective, so that an optimal allocation rule for the ironed virtual valuations is equal to the optimal monotone allocation rule for the original virtual valuations.

Given a distribution function  $F(\cdot)$  with virtual valuation function  $\vartheta(\cdot)$ , the *ironed virtual valuation function*,  $\bar{\vartheta}(\cdot)$ , is constructed as follows:

- 1) For  $q \in [0, 1]$ , define  $h(q) = \vartheta(F^{-1}(q))$ .
- 2) Define  $H(q) = \int_0^q h(r)dr$ .
- 3) Define  $G$  as the convex hull of  $H$  — the largest convex function bounded above by  $H$  for all  $q \in [0, 1]$ .
- 4) Define  $g(q)$  as the derivative of  $G(q)$ , where defined, extended to all of  $[0, 1]$  by right-continuity.
- 5) Finally, define  $\bar{\vartheta}(z) = g(F(z))$ .

Convexity of  $G$  implies that Step 4 of the ironing procedure is well defined and that  $g$ , and hence  $\bar{\vartheta}$ , is a monotone non-decreasing function.

From the main theorem of Myerson (1981), maximizing the expectation of a general linear objective subject to incentive compatibility is equivalent

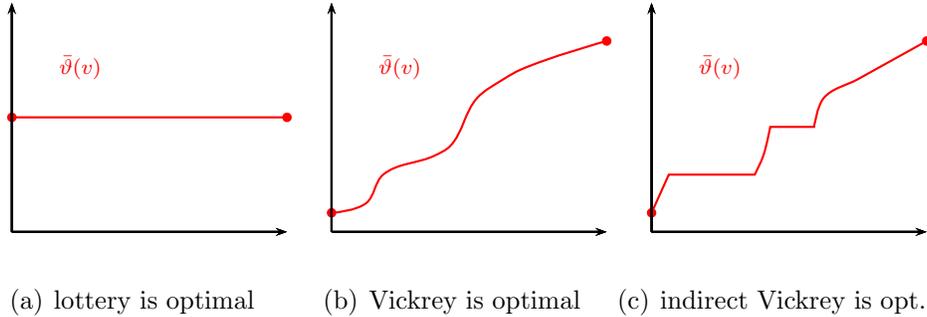


FIGURE 1. IRONED VIRTUAL VALUE FUNCTIONS IN THE THREE DISTRIBUTIONAL CASES. FOR THE OBJECTIVE OF CONSUMER SURPLUS THE CASES CORRESPOND TO (A) MHR DISTRIBUTIONS, (B) ANTI-MHR DISTRIBUTIONS, AND (C) NON-MHR DISTRIBUTIONS.

to maximizing the expected ironed virtual surplus. Different tie-breaking rules, however, can yield different optimal mechanisms. In our symmetric settings, with i.i.d. agents and the symmetric feasibility constraint  $\mathcal{X}$  of  $k$ -unit auctions, it is natural to consider symmetric optimal mechanisms.

**THEOREM 2:** *For every general linear objective and distribution  $F$ , the  $k$ -unit auction that allocates the units to the agents with the highest non-negative ironed virtual values, breaking ties randomly and discarding all left-over units, maximizes the expected value of the objective.*

#### INTERPRETATION FOR CONSUMER SURPLUS MAXIMIZATION.

Consider the consumer surplus objective, where  $\vartheta(v) = \frac{1-F(v)}{f(v)}$ , and the following three types of distributions (Figure 1). Monotone hazard rate (MHR) distributions; e.g., uniform, normal, and exponential; have monotone non-increasing  $\vartheta(v)$ . In this case, ironing  $\vartheta(\cdot)$  to be non-decreasing results in  $\bar{\vartheta}(\cdot) = \mathbf{E}[v]$ , a constant function. The optimal (symmetric) mechanism is therefore a lottery that awards the  $k$  units to  $k$  agents uniformly at random.

For distributions with a hazard rate monotone in the opposite direction, henceforth *anti-MHR* distributions,  $\vartheta(\cdot)$  is non-negative and monotone non-decreasing. Power-law distributions, such as  $F(z) = 1 - 1/z^c$  with  $c > 0$  on  $[1, \infty)$ , are canonical examples. In this case, the optimal mechanism awards the  $k$  units to the  $k$  highest valued agents, i.e., it is the  $k$ -Vickrey auction. Thus, as also observed by McAfee and McMillan (1992), Chakravarty and Kaplan (2006), and Condorelli (2007), the optimal mechanism depends on whether or not the distribution is heavy-tailed.

The final case occurs when the distribution is neither MHR nor anti-MHR, henceforth *non-MHR*. Here, the ironed virtual valuation function  $\bar{\vartheta}(\cdot)$  is constant on some intervals and monotone increasing on other intervals. The optimal mechanism can be described, for instance, as an indirect Vickrey auction where agents are not allowed to bid on intervals where the ironed virtual value is constant. For example, consider the two-point distribution with probability mass  $\frac{1}{2}$  on 1 and  $\frac{1}{2}$  on  $h > 1$ . Provided  $h$  is sufficiently large, the consumer-surplus-maximizing mechanism allocates to a random high-value agent or, if there are no high-value agents, to a random (low-value) agent. This final case is the most general, in that it subsumes both the MHR and anti-MHR cases. Our general theory of platform design necessitates understanding this non-MHR case in detail.

### III. The Performance Benchmark

In this section we leverage the characterization of Bayesian optimal mechanisms from the preceding section to identify and characterize a simple prior-free performance benchmark. This constitutes the second step of our approach to platform design.

The performance benchmark is derived as follows. As discussed in Section II, Bayesian optimal mechanisms are ironed virtual surplus optimizers. For  $k$ -unit environments, these mechanisms simply select the  $k$  agents with the highest non-negative ironed virtual values. Among these optimal mechanisms, the symmetric one breaks ties randomly. Denote the symmetric optimal mechanism for distribution  $F$  by  $\text{Opt}_F$ . Denote by  $\text{Opt}_F(\mathbf{v})$  the expected performance (over the choice of random allocation) obtained by the mechanism  $\text{Opt}_F$  on the valuation profile  $\mathbf{v}$ .

DEFINITION 1: *The performance benchmark is the supremum of Bayesian optimal mechanisms,  $\mathcal{G}(\mathbf{v}) = \sup_F \text{Opt}_F(\mathbf{v})$ .*

For one interpretation of the definition of  $\mathcal{G}$ , observe that

$$(4) \quad \mathbf{E}_{\mathbf{v}}[\mathcal{G}(\mathbf{v})] \geq \mathbf{E}_{\mathbf{v}}[\text{Opt}_F(\mathbf{v})]$$

for valuation profiles drawn i.i.d. from an arbitrary distribution  $F$ . Thus, the approximation of the performance benchmark  $\mathcal{G}$  implies the simultaneous approximation of all symmetric Bayesian optimal mechanisms.

We now give a simple characterization of the performance benchmark for general linear objectives by considering ex post outcomes of symmetric Bayesian optimal mechanisms. When  $k$  units are available, a symmetric Bayesian optimal mechanism serves these units to the  $k$  agents with the highest non-negative ironed virtual values. Ties, which occur in ironed virtual surplus maximization when two (or more) agents' values are mapped to same ironed virtual value, are broken randomly. Ex post, we can classify the agents into at most three groups: those that win with certainty (winners), those that lose with certainty (losers), and those that win with a common

probability strictly between 0 and 1 (partial winners).

DEFINITION 2: A two-level  $(p, q)$ -lottery, denoted  $\text{Lot}_{p,q}$ , first serves agents with values strictly more than  $p$ , then serves agents with values strictly more than  $q$ , while supplies last (breaking ties randomly, as needed). All agents with values at most  $q$  are rejected.

It will be useful to calculate explicitly, using Lemma 1, the payments of a two-level lottery. Let  $S$  and  $T$  denote the sets of agents with value in the ranges  $(p, \infty)$  and  $(q, p]$ , respectively. Let  $s = |S|$  and  $t = |T|$ . For simplicity, assume that  $s \leq k < s + t$ , where  $k$  is the number of units available. The payments are as follows.

- 1) Agents  $i \in S$  are each allocated a unit and charged

$$(5) \quad p_i = p - (p - q) \frac{k-s+1}{t+1}.$$

- 2) The remaining  $k - s$  units are allocated uniformly at random to the  $k - s$  agents  $i \in T$ , i.e., by lottery; each such winner pays  $p_i = q$ .

We characterize the performance benchmark for platform design for general linear objectives in terms of two-level lotteries.

THEOREM 3: *The supremum of Bayesian optimal mechanisms benchmark satisfies  $\mathcal{G}(\mathbf{v}) = \sup_F \text{Opt}_F(\mathbf{v}) = \sup_{p,q} \text{Lot}_{p,q}(\mathbf{v})$ .*

PROOF: The outcome of ironed virtual surplus maximization is equivalent to a  $k$ -unit  $(p, q)$ -lottery. To see this, consider an ironed virtual valuation function  $\bar{v}$  and a valuation profile  $\mathbf{v}$ . Set  $p$  to be the infimum bid that the highest-valued agent can make and be a winner (possibly larger than the

agent’s value), and  $q$  to be the infimum bid that a partial winner can make and remain a partial winner (or  $p$  if there are no partial winners). The two mechanisms have the same outcome on profile  $\mathbf{v}$ . Conversely, every  $(p, q)$ -lottery arises in ironed virtual surplus maximization with respect to some i.i.d. distribution, for example with  $\bar{v}(v) = 2$  for  $v \in (p, \infty)$ ,  $\bar{v}(v) = 1$  for  $v \in (q, p]$ , and  $\bar{v}(v) = -1$  for  $v \leq q$ .<sup>4</sup> Q.E.D.

We conclude with a simple but useful observation: The values of  $p$  and  $q$  that attain the supremum in Theorem 3 must each either be zero, infinity, or an agent’s value. Observe that the objective  $\sum_i \gamma_v v_i x_i + \gamma_p p_i$  is linear in payments. If  $q$  or  $p$  is not in the valuation profile, then it can either be increased or decreased without decreasing the objective. For example, lowering  $p$  or  $q$  without changing the allocation increases consumer surplus.

#### IV. Consumer Surplus

In this section we consider platform design for the objective of consumer surplus. We consider separately the  $n = 2$  agent case and the general  $n > 2$  agent case. For  $n = 2$  agents (and a single unit) we completely execute our template for platform design by reinterpreting the benchmark, giving a platform mechanism that is universally adopted with competitive advantage  $4/3$ , and proving that no platform mechanism is universally adopted with a smaller competitive advantage. The platform mechanism that achieves this bound is neither a standard auction nor a mixture over standard auctions, where by “standard” we mean a symmetric Bayesian-optimal mechanism with respect to some i.i.d. valuation distribution.

For every number  $n > 2$  of agents and  $k \geq 1$  of items, we give a heuristic

<sup>4</sup>For objectives like consumer surplus where the virtual values are always non-negative, set  $\bar{v}(v) = 1/2$  instead of  $-1$  for  $v \leq q$ . See the construction in Appendix A for details.

platform that guarantees universal adoption with a constant competitive advantage (independent of  $k$ ,  $n$ , and the support of the valuations). This platform is not a mixture of standard auctions, and we show that no such mixture is universally adopted with any finite competitive advantage (as  $n \rightarrow \infty$ ). This heuristic mechanism identifies properties of good platforms and is a proof-of-concept that good platforms exist.

### A. *Single-unit Two-agent Platforms*

We now execute the framework for platform design for two agents, a single unit, and the objective of consumer surplus. Bayesian optimal mechanisms and our benchmark are characterized in Sections II and III, respectively; for two agents and a single item, the benchmark takes a simple form.

There are only two relevant  $(p, q)$ -lotteries for the performance benchmark, the degenerate  $p = q = 0$  lottery, and the  $p = v_{(2)}$  and  $q = 0$  lottery; here  $v_{(1)}$  and  $v_{(2)}$  denote the highest and second-highest agent values, respectively. From equation (5), the consumer surpluses of these two-level lotteries are  $\frac{v_1+v_2}{2}$  (i.e., the average value) and  $v_{(1)} - \frac{v_{(2)}}{2}$ , respectively. Thus,

$$(6) \quad \mathcal{G}(\mathbf{v}) = \max\left\{\frac{v_1+v_2}{2}, v_{(1)} - \frac{v_{(2)}}{2}\right\}.$$

This benchmark is depicted in Figure 2(a).

We now turn to the problem of designing a platform mechanism that is universally adopted with a minimal competitive advantage. As mentioned above, the lottery is adopted with a competitive advantage of 2. A natural approach to platform design is to randomly mix over two platforms that are good in different settings. For example, the Vickrey auction is good on the valuation profile  $\mathbf{v} = (1, 0)$ , whereas the lottery is good on the

valuation profile  $\mathbf{v} = (1, 1)$ . Considering only these two valuation profiles (where  $\mathcal{G}(\mathbf{v}) = 1$ ), choosing the Vickrey auction with probability  $1/3$  and the lottery with probability  $2/3$  balances the competitive advantage necessary for adoption of the platform for each profile at  $3/2$ . In fact, a routine calculation shows that this mixture is universally adopted with competitive advantage  $3/2$ . This platform mechanism, however, is not optimal.

One approach to solving for the optimal platform mechanism is to look for a mechanism that achieves the same approximation factor to the benchmark for every valuation profile.<sup>5</sup> Inspecting the benchmark (Figure 2(a)), we conclude that an auction with identical approximation factor on all inputs must have a discontinuity in its outcome only where the ratio between the high and low value is 2. Importantly, there should be no discontinuity in its outcome when the values are equal, that is, the optimal platform should never mix over the Vickrey auction. These observations suggest the following parameterized class of auctions.

**DEFINITION 3:** *The two-agent single-item ratio auction with ratio  $\alpha \geq 1$  and bias  $\chi \in [1/2, 1]$  allocates the good according to a fair coin if the agent values are within a factor  $\alpha$  of each other and, otherwise, according to a biased coin with probability  $\chi$  in favor of the high-value agent.*<sup>6</sup>

The Vickrey auction and the lottery are special cases of the ratio auction. With bias  $1/2$  the ratio auction is a lottery (for every ratio); with ratio  $\alpha = 1$  and bias  $\chi = 1$  it is the Vickrey auction. We next show that the optimal two-agent single-item platform for consumer surplus is the ratio auction with ratio  $\alpha = 2$  and bias  $\chi = 3/4$ . The allocation probabilities

<sup>5</sup>The optimal platform for monopoly pricing from Section I also exhibits this property.

<sup>6</sup>Appropriate payments can be derived by reinterpreting the ratio auction as a distribution over weighted Vickrey auctions; see also the proof of Lemma 3.

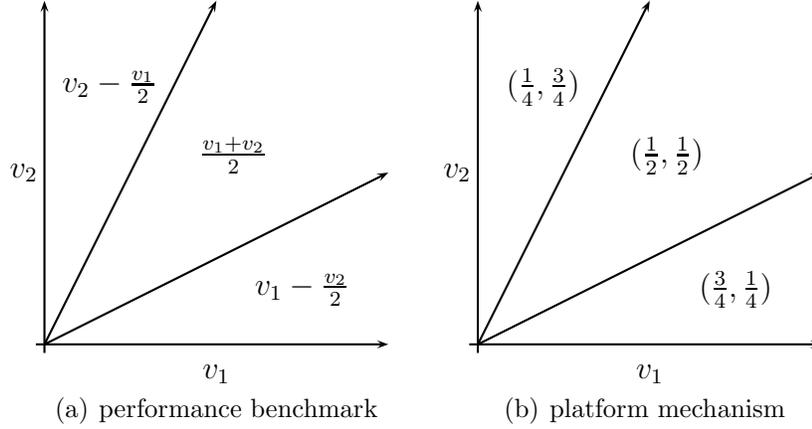


FIGURE 2. THE PERFORMANCE BENCHMARK (6) AND OPTIMAL PLATFORM MECHANISM FOR THE SINGLE-ITEM, TWO-AGENT, CONSUMER-SURPLUS-MAXIMIZATION PROBLEM. THE POSITIVE QUADRANT IS PARTITIONED BY THE LINES  $v_1 = 2v_2$  AND  $2v_1 = v_2$ . THE ALLOCATION RULE OF THE PLATFORM MECHANISM IS GIVEN AS  $(x_1, x_2)$ .

of this auction are depicted in Figure 2(b). It is adopted with competitive advantage  $4/3$ .

LEMMA 3: *The ratio auction with ratio  $\alpha = 2$  and bias  $\chi = 3/4$  is universally adopted with competitive advantage  $4/3$ .*

PROOF: The ratio auction (with ratio  $\alpha$ ) can always be expressed as a distribution over weighted Vickrey auctions, where  $w_1 = 1$ ,  $w_2$  is selected randomly from some distribution over the set  $\{0, 1/\alpha, \alpha, \infty\}$ , and the agent  $i$  that maximizes  $w_i v_i$  winning the item. With bias  $\chi = 3/4$ , the distribution over the set is uniform. We calculate the auction's approximation of the benchmark via simple case analysis. The expected consumer surplus from the four choices of  $w_2$  averages, when  $v_1 \in [v_2/2, 2v_2]$ , to

$$\frac{1}{4} \left[ v_1 + \left( v_1 - \frac{v_2}{2} \right) + \left( v_2 - \frac{v_1}{2} \right) + v_2 \right] = \frac{3}{4} \frac{v_1 + v_2}{2}$$

and, when  $v_1 > 2v_2$ , to

$$\frac{1}{4} \left[ v_1 + \left( v_1 - \frac{v_2}{2} \right) + (v_1 - 2v_2) + v_2 \right] = \frac{3}{4} \left( v_1 - \frac{v_2}{2} \right).$$

The case where  $v_1 < v_2/2$  is symmetric. In each case, the expected consumer surplus is exactly  $\frac{3}{4}\mathcal{G}(\mathbf{v})$ . Q.E.D.

We now show that the ratio auction with ratio  $\alpha = 2$  and bias  $\chi = 3/4$  is an optimal platform; meaning, no platform is universally adopted with competitive advantage less than  $4/3$ . We first note that, for every distribution  $F$ , the expected consumer surplus of the ratio auction with ratio  $\alpha = 2$  and bias  $\chi = 3/4$  is exactly  $3/4$  times the expected value of the benchmark  $\mathcal{G}$ . Of course, the Bayesian optimal auction for  $F$  is no worse.

**COROLLARY 1:** *For every distribution  $F$  and  $n = 2$  agents and  $k = 1$  item, the expected benchmark is at most  $4/3$  times the expected consumer surplus of the optimal auction, that is,  $\mathbf{E}[\mathcal{G}(\mathbf{v})] \leq \frac{4}{3}\mathbf{E}[\text{Opt}_F(\mathbf{v})]$ .*

The following technical lemma exhibits a distribution  $F$  for which the inequality in Corollary 1 is tight. Intuitively, this distribution is the one with constant virtual value for utility.

**LEMMA 4:** *For the exponential distribution  $F(z) = 1 - e^{-z}$ ,  $n = 2$  agents,  $k = 1$  unit, the expected value of the benchmark is  $4/3$  times the expected consumer surplus of the optimal auction, that is,  $\mathbf{E}[\mathcal{G}(\mathbf{v})] = \frac{4}{3}\mathbf{E}[\text{Opt}_F(\mathbf{v})]$ .*

**PROOF:** Since the exponential distribution has a monotone hazard rate, a lottery maximizes the expected consumer surplus (Section II). The expected value of an exponential random variable is 1 so  $\mathbf{E}[\text{Opt}_F(\mathbf{v})] = \mathbf{E}[v] = 1$ .

We now calculate the expected value of the benchmark  $\mathcal{G}(\mathbf{v})$  defined in equation (6). Write the smaller value as  $v = v_{(2)}$  and the higher value as

$x + v = v_{(1)}$  for  $x \geq 0$ . In terms of  $v$  and  $x$  the benchmark is  $v + \frac{x}{2}$  when  $x \leq v$  and  $\frac{v}{2} + x$  when  $x \geq v$ . Therefore, the expectation of  $\mathcal{G}$  conditioned on  $v$  is

$$\begin{aligned} \mathbf{E}[\mathcal{G}(x + v, v) \mid v] &= \int_0^v \left(v + \frac{x}{2}\right) e^{-x} dx + \int_v^\infty \left(\frac{v}{2} + x\right) e^{-x} dx \\ &= v(1 - e^{-v}) + \frac{1}{2}(1 - (v + 1)e^{-v}) + \frac{v}{2}e^{-v} + (v + 1)e^{-v} \\ &= v + \frac{1}{2}(1 + e^{-v}). \end{aligned}$$

The smaller value  $v_{(2)} = v$  is distributed according to an exponential distribution with rate 2. Integrating out yields

$$\begin{aligned} \mathbf{E}[\mathcal{G}(x + v, v)] &= \int_0^\infty \left(v + \frac{1}{2} + \frac{1}{2}e^{-v}\right) 2e^{-2v} dv \\ &= \frac{1}{2} + \frac{1}{2} + \int_0^\infty e^{-3v} dv = \frac{4}{3}. \end{aligned}$$

Q.E.D.

For the setting of Lemma 4, the optimal mechanism has expected consumer surplus  $\frac{3}{4}\mathbf{E}[\mathcal{G}(\mathbf{v})]$ . Any platform mechanism is only worse and, by the definition of expectation, there must be a valuation profile  $\mathbf{v}$  where this platform mechanism has consumer surplus at most  $\frac{3}{4}\mathcal{G}(\mathbf{v})$ .

**COROLLARY 2:** *For  $n \geq 2$  agents,  $k = 1$  item, and the consumer surplus objective, no platform mechanism is universally adopted with competitive advantage less than  $4/3$ .*

We conclude that the ratio auction with ratio  $\alpha = 2$  and bias  $\chi = 3/4$  is an optimal platform for two-agent, single-item consumer surplus maximization.

*B. Multi-agent Platforms: Standard Mechanisms Are Not Universally Adopted*

For markets with  $n > 2$  agents, neither the Vickrey auction, the lottery, nor a convex combination thereof is universally adopted with a constant competitive advantage. For instance, with  $k = 1$  unit and valuation profile  $\mathbf{v} = (1, 1, 0, \dots, 0)$ , the Vickrey auction has zero consumer surplus and the lottery has expected consumer surplus  $2/n$ , while the benchmark consumer surplus is  $\mathcal{G}(\mathbf{v}) = 1$  (Definition 1). In fact, no Bayesian optimal auction (a.k.a., standard auction) or mixture over standard auctions is universally adopted either. Consequentially, as will be described in Section IV.C, the derivation of a platform mechanism that is universally adopted with a constant competitive advantage requires non-standard auction designs. The proof of the following theorem is in Appendix B.

**THEOREM 4:** *For every  $\rho > 1$  there is a sufficiently large  $n$  such that, for an  $n$ -agent, 1-unit setting, no mixture over standard auctions is universally adopted with competitive advantage  $\rho$ .*

The intuition for the theorem comes from viewing the problem as a zero-sum game of hide and seek between the platform designer (the seeker) and nature (the hider) with a large number  $\beta$  of locations. If the seeker finds the hider (i.e., they choose the same location  $\kappa \in \{0, \dots, \beta - 1\}$ ) then the seeker's payoff is about  $\beta$ . Otherwise, the hider evades the seeker and the seeker's payoff is about 1. The value of this game (for the seeker) is about 2 and is given by the unique equilibrium where both the hider and seeker picking locations uniformly at random.

To relate this hide-and-seek game back to the platform design problem, consider the following actions of the designer and nature. Nature's actions are to pick one of  $\beta$  value distributions where the  $\kappa$ th distribution is constructed to have virtual value  $\beta$  on interval  $[\kappa\beta, \kappa\beta + \beta]$  and virtual value 1 everywhere else. Such a distribution can be constructed as a piece-wise exponential distribution as described in detail in Appendix A. The platform designer's action will be to pick one of  $\beta$  mechanisms where mechanism  $\kappa$  is highest-bid-wins with ironing on  $[\kappa\beta, \infty)$ , i.e., the optimal mechanism for the  $\kappa$ th distribution.

With sufficiently many agents (specifically  $n > e^{\beta^2}$ ) the designers payoffs are as follows. If the designer and nature pick corresponding actions, the designer's payoff is about  $\beta$ . This follows as with high probability the winning agent has virtual value  $\beta$ . If the designer and nature pick non-corresponding actions then the designer's payoff is about 1 as with high probability the winning agent has virtual value 1. (These high probability results follow because the constructed distributions are piece-wise exponential.)

From this hide-and-seek analogy we see that the platform designer's payoff is a constant, i.e., about 2, while for any of nature's distributions the optimal consumer surplus is about  $\beta$ , an arbitrarily large number. We conclude that no randomization over these Bayesian optimal mechanisms is universally adopted with a constant competitive advantage. To extend the above argument to prove Theorem 4, it remains to generalize to all mixtures over standard auctions not just the ones in the hide-and-seek analogy. These details are deferred to the formal proof in Appendix B.

*C. Multi-unit Multi-agent Platforms: A Universally Adopted Platform*

We now upper bound the minimum competitive advantage for universal adoption by an absolute constant; this upper bound is independent of the number of units, the number of agents, and the support size of the valuations. In contrast to the preceding section, this bound shows the existence of good platform mechanisms. To make this task analytically tractable we relax the problem of identifying the optimal platform and instead look for a simple heuristic platform that is universally adopted with a constant competitive advantage. Neither is the mechanism we identify the best possible, nor is our analysis of it tight; however, the simplicity of the heuristic mechanism and our analysis of its performance allows the main features that go into good platform mechanisms to be identified and interpreted. In contrast, the optimal platform mechanism, even if it could be identified, is likely to be too complex to interpret.

The heuristic mechanism is based on the random sampling paradigm of Goldberg et al. (2001). Half of the agents (henceforth: the sample) are used for a market analysis to determine a good mechanism to run on the other half of the agents (henceforth: the market). The family of good mechanisms that we will consider are one-level lotteries (below, Definition 4). Importantly, the resulting random-sampling-based mechanism (below, Definition 5) does not use the sample to explicitly estimate the distribution of agent preferences. Moreover, we have deliberately avoided optimizing the parameters of the mechanism in order to keep its description and analysis as simple as possible.

**DEFINITION 4:** *The one-level  $r$ -lottery, denoted  $\text{Lot}_r$ , serves agents with values strictly more than  $r$ , while supplies last (breaking ties randomly).*

*Winners are charged  $r$  and agents with values below  $r$  are rejected.*

DEFINITION 5: *The  $k$ -unit Random Sampling Optimal Lottery (RSOL) mechanism works as follows.*

- 1) *Partition the agents uniformly at random into a market  $M$  and a sample  $S$ , i.e., each agent is in  $S$  or  $M$  independently with probability  $1/2$  each.*
- 2) *Calculate the optimal  $k$ -unit lottery price  $r_S$  for the sample:  $r_S = \operatorname{argmax}_r \operatorname{Lot}_r(\mathbf{v}_S)$ .*
- 3) *Run the  $k$ -unit  $r_S$ -lottery on the market  $M$ ; reject the agents in the sample  $S$ .*

It is easy to see that RSOL is dominant strategy incentive compatible. A one-level lottery at any fixed price is incentive compatible, and the agents in the market face a one-level lottery with price set by the agents in the sample.

The performance analysis of RSOL consists of two main steps. First, we show that the performance of the optimal one-level lotteries (as used by the mechanism) is within a factor of two of the performance of the optimal two-level lottery (i.e., the benchmark). Second, we show that either RSOL or the  $k$ -unit Vickrey auction has good consumer surplus. The probabilistic analysis of RSOL shows that the one-level lottery chosen for the market has expected consumer surplus close to the consumer surplus of the ex post optimal one-level lottery on the full valuation profile. This result is stated as Theorem 5, below, and formally proved in Appendix C.

THEOREM 5: *For every  $n, k \geq 1$ , there is an  $n$ -agent  $k$ -unit platform mechanism that is universally adopted with constant competitive advantage.*

The mechanism in the statement of the theorem, as alluded to above, is a convex combination of RSOL and the  $k$ -unit Vickrey auction, i.e., the auction that sells to the top  $k$  valued agents at the  $k + 1$ st price. The reason for this combination is that RSOL does not get good consumer surplus when most of the optimal consumer surplus comes from the agent with the highest value. For example, for the  $n$ -agent valuation profile  $\mathbf{v} = (1, \epsilon, \dots, \epsilon)$  and one unit, RSOL's expected consumer surplus is about  $1/n + \epsilon$  while the optimal consumer surplus is  $1 - \epsilon$ .

## V. Platform Design and Prior-Free Profit Maximization

While the objective of profit maximization is not central to this paper, there have been a number of studies of prior-free mechanisms for profit maximization that are relevant to platform design. This section discusses digital good settings (Section V.A), multi-unit settings (Section V.B), and more general settings (Section V.C). We describe these results using the terminology of platform design. An important goal of our discussion is to compare our performance benchmark, which is justified by Bayesian foundations, with the prior-free benchmarks employed in this literature.

### A. Digital Good Settings

The simplest setting for platform design is that of a *digital good*, i.e., a multi-unit setting with the same number  $k = n$  of units as (unit-demand) agents. This environment admits a trivial optimal mechanism for surplus and consumer surplus (serve all agents for free); but for profit maximization, designing a good platform mechanism is a challenging problem.

The Bayesian optimal mechanism for a digital good when values are drawn i.i.d. from the distribution  $F$  simply posts the monopoly price for  $F$ , i.e.,

an  $r$  that maximizes  $r(1 - F(r))$ . In the language of the preceding sections, this optimal mechanism can be viewed as an  $r$ -lottery. The performance benchmark described in Section III simplifies to

$$(7) \quad \mathcal{G}(\mathbf{v}) = \max_i iv_{(i)}.$$

For  $n = 1$  agent, the benchmark (7) equals the surplus and, as we concluded in Section I, it cannot be well approximated by any platform mechanism. Because of this technicality, the benchmark  $\mathcal{G}^{(2)}$  to which prior-free digital good auctions have been compared (e.g., Goldberg et al., 2006) explicitly excludes the possibility of deriving all its profit from one agent:

$$(8) \quad \mathcal{G}^{(2)}(\mathbf{v}) = \max_{i \geq 2} iv_{(i)}.$$

Therefore, up to the technical difference between benchmarks (7) and (8), the prior-free literature for digital goods is compatible with our framework for platform design. Some notable results in this literature are as follows. For reasons we explain shortly, we refer to the approximation of  $\mathcal{G}^{(2)}$  as giving near-universal adoption. Optimal platform mechanisms are given in Goldberg et al. (2006) and Hartline and McGrew (2005) for two and three-player digital goods settings, where the competitive advantages for near-universal adoption are precisely 2 and  $13/6$ , respectively. As the number  $n$  of agents tends to infinity, Goldberg et al. (2006) show that there is no platform mechanism that is near-universally adopted with competitive advantage less than 2.42; and Chen et al. (2014) show that there exists a mechanism that matches this bound. This optimal platform mechanism is fairly complex; Hartline and McGrew (2005) had previously given a simple mechanism that

is near-universally adopted with competitive advantage 3.25.

The benchmark  $\mathcal{G}^{(2)}$  does not satisfy our most basic requirement for benchmarks: there exist distributions for which the expected Bayesian optimal revenue exceeds the expected value of the benchmark.<sup>7</sup> Therefore, mechanisms that approximate the benchmark may not be universally adopted. This problem is not an artifact of the  $\mathcal{G}^{(2)}$  benchmark and is inherent to profit maximization: pathological distributions show that there is *no* benchmark  $\mathcal{G}'$  and constant  $\beta$  that satisfy  $\beta \mathbf{E}_{\mathbf{v}}[\text{Opt}_F(\mathbf{v})] \geq \mathbf{E}_{\mathbf{v}}[\mathcal{G}'(\mathbf{v})] \geq \mathbf{E}_{\mathbf{v}}[\text{Opt}_F(\mathbf{v})]$  for every distribution  $F$ .

For the profit objective, the requirement of universal adoption can be relaxed to adoption for every distribution in a large permissive class of distributions. Approximation of the benchmark  $\mathcal{G}^{(2)}$  implies such a near-universal adoption, in the following sense.

**PROPOSITION 1:** *If mechanism  $\mathcal{M}$  is a  $\beta$  approximation to  $\mathcal{G}^{(2)}$  on all valuation profiles, i.e.,  $\mathcal{M}(\mathbf{v}) \geq \mathcal{G}^{(2)}(\mathbf{v})/\beta$  for every  $\mathbf{v}$ , then  $\mathcal{M}$  is adopted with competitive advantage  $\beta$  on distributions  $F$  with  $\mathbf{E}_{\mathbf{v}}[\mathcal{G}^{(2)}(\mathbf{v})] \geq \mathbf{E}_{\mathbf{v}}[\text{Opt}_F(\mathbf{v})]$ .*

Proposition 1 has bite in that it is satisfied by most relevant distributions. The following lemma, which we prove in Appendix D, gives a sufficient condition for the distribution. Intuitively, this condition states that the revenue from posting a price does not drop too quickly as that price is lowered, and it is a strict generalization of the regularity condition of Myerson (1981). This condition is not satisfied in the bad example above, as most of the optimal revenue is derived from one high-valued agent.

<sup>7</sup>For example, consider  $n$  agents, each having value  $n^2$  with probability  $1/n^2$  and 0 otherwise. The expected revenue of a Bayesian optimal mechanism is  $n$ . The expected value of the benchmark  $\mathcal{G}^{(2)}$  is bounded above by a constant, independent of  $n$ .

LEMMA 5: *For digital good settings and every distribution  $F$  with  $v(1 - F(v))/F(v)$  non-increasing,  $\mathbf{E}_{\mathbf{v}}[\mathcal{G}^{(2)}(\mathbf{v})] \geq \mathbf{E}_{\mathbf{v}}[\text{Opt}_F(\mathbf{v})]$ .*

### B. Multi-unit Settings

We next consider maximizing profit in a  $k$ -unit auction with unit-demand bidders. We assume throughout that  $k \geq 2$ . We next define a variant of the performance benchmark  $\mathcal{G}^{(2)}$  for platform design and compare it to the benchmark  $\mathcal{F}^{(2)}(\mathbf{v}) = \max_{2 \leq i \leq k} iv_{(i)}$  that has been employed, without formal justification, in previous work on prior-free multi-unit auctions.

The benchmark  $\mathcal{G}$  defined as the supremum of Bayesian optimal mechanisms is, by Theorem 3, equivalent to the supremum over two-level lotteries (which need not sell all units). Two-level lotteries are not useful for profit-maximization in digital goods settings, where a  $(p, q)$ -lottery is equivalent to a  $q$ -lottery which is equivalent to a  $q$  price posting. They are useful in limited supply settings, however. The performance benchmark  $\mathcal{G}^{(2)}$  is defined by  $\mathcal{G}^{(2)}(\mathbf{v}) = \mathcal{G}(v_{(2)}, v_{(2)}, v_{(3)}, \dots, v_{(n)})$ . For every valuation profile, the benchmark  $\mathcal{G}^{(2)}$  is at most twice the value of  $\mathcal{F}^{(2)}$  (cf. Lemma 6). Thus, every multi-unit auction that  $\beta$ -approximates the benchmark  $\mathcal{F}^{(2)}$  also  $2\beta$ -approximates the benchmark  $\mathcal{G}^{(2)}$ . As in Section V.A, approximation of the benchmark  $\mathcal{G}^{(2)}$  implies approximation of the optimal expected revenue in every Bayesian setting with a non-pathological distribution.

The above discussion provides Bayesian foundations for the benchmark  $\mathcal{F}^{(2)}$ , and translates the known results for approximating that benchmark into good platform designs. Specifically, every digital good auction that  $\beta$ -approximates the  $\mathcal{F}^{(2)}$  benchmark — equivalently for a digital good, the  $\mathcal{G}^{(2)}$  benchmark — can be easily converted into a multi-unit auction that

$\beta$ -approximates the  $\mathcal{F}^{(2)}$  benchmark (Goldberg et al., 2006) and hence  $2\beta$ -approximates the  $\mathcal{G}^{(2)}$  benchmark. The optimal platform mechanism for digital goods from Chen et al. (2014) can be thus converted into a platform mechanism for limited supply that is near-universally adopted with competitive advantage 4.84, i.e., it is a 4.84-approximation to  $\mathcal{G}^{(2)}$ .

### C. General Environments

The approach to Bayesian optimal mechanism design discussed in Section II characterizes optimal mechanisms beyond just multi-unit settings. For every single-parameter setting, where agents want service and there is a feasibility constraint over the set of agents that can be simultaneously served, the optimal mechanism is the ironed virtual surplus maximizer.

Our benchmark  $\mathcal{G}$  is difficult to analyze beyond multi-unit settings. In follow-up work to this paper, Hartline and Yan (2011) gave a refinement of our benchmark using the notion of envy-freedom. For instance, when the set system that constrains feasible outcomes satisfies a substitutes condition (formally: the set of feasible outcomes are the independent sets of a matroid set system), their benchmark is at least as large as ours. They give a mechanism that is similar to the RSOL (Definition 5) that, for these set systems, is near-universally adopted with constant competitive advantage.

## VI. Discussion

We defined an analysis framework for platform design based on relative approximation of a performance benchmark. Auctions that approximate this benchmark are simultaneously near-optimal in every Bayesian setting with i.i.d. bidder valuations. Optimizing within this analysis framework suggests novel multi-unit auction formats, different from those suggested by

Bayesian analysis. The framework is flexible and permits several extensions and modifications, discussed next.

We focused on platform design for the objectives of consumer surplus (Section IV) and profit maximization (Section V), but our platform design approach extends beyond these objectives. As an example, imagine the  $k$ -unit auction in an i.i.d. Bayesian setting where the optimal solution is characterized by optimizing the ironed virtual value corresponding to “a 8% government sales tax.” The objective is then the value of the agents and mechanism less the tax deducted by government, and the corresponding virtual value function has the form  $\varphi(v) = 0.92v - 0.08\frac{1-F(v)}{f(v)}$ . The optimal  $k$ -unit  $(p, q)$ -priority lottery remains the appropriate benchmark for this and every other linear objective. For every linear objective with  $\gamma_p < 0$ , see equation (1), there is always an optimal  $(p, q)$ -priority lottery with  $p$  and  $q$  at most the second highest value  $v_{(2)}$ , and the mechanism RSOL of Section IV.C can be mixed with a Vickrey auction to approximate the benchmark.

Optimal mechanisms are ironed virtual surplus maximizers in every single-parameter Bayesian setting with independent private values (Myerson, 1981), not just in the multi-unit auction settings studied here. Examples of more general single-parameter settings include constrained matching markets, single-minded combinatorial auctions, and public projects. The performance benchmark (Definition 1) can again be defined pointwise as the supremum over the performance of ironed virtual surplus maximizers on a given valuation profile. As discussed in Section V.C, this performance benchmark seems hard to characterize beyond multi-unit settings (cf., Theorem 3). The follow-up work of Hartline and Yan (2011) recently proposed an alternative

benchmark based on envy freedom. This benchmark has structure similar to that of Bayesian optimal auctions and, for this reason, is analytically tractable. Hartline and Yan (2011), for the profit objective and the envy-free benchmark, give platform mechanisms that are near-universally adopted with constant competitive advantage in general settings.

We defined the performance benchmark as the supremum performance of (symmetric) mechanisms that are optimal in some Bayesian setting with i.i.d. valuations. Clearly, one can define such a benchmark with respect to *any* class of mechanisms. As an example alternative, consider consumer surplus maximization and the class of symmetric mechanisms that are Bayesian optimal for some i.i.d. distribution that is either MHR or anti-MHR — that is, the class consisting solely of the Vickrey auction and the (zero-price) lottery. This class arises naturally when domain knowledge suggests that only MHR and anti-MHR distributions are relevant, or if outside consultants are only equipped to design optimal mechanisms for these cases. Specializing to the two-bidder single-item case studied in Section IV.A, the platform design benchmark decreases from  $\max\{\frac{v_1+v_2}{2}, v_{(1)} - \frac{v_{(2)}}{2}\}$  to  $\max\{\frac{v_1+v_2}{2}, v_{(1)} - v_{(2)}\}$ . Reworking the analysis of that section for this new benchmark shows that the optimal mechanism remains a ratio auction, just with a different setting of the parameters (namely,  $\alpha = 3$  and  $\chi = 4/5$ , for an approximation ratio of  $5/4$ ). Notably, the format of the optimal platform is robust to this particular change in the benchmark.

The mechanism in Section IV.C demonstrates the existence of platform mechanisms for consumer surplus maximization that are universally adopted with constant competitive advantage, independent of the number of units and bidders. No standard auction, meaning an ironed virtual surplus maxi-

mizer, enjoys such a guarantee (Section IV.B). Developing our understanding of  $n$ -player platform design further is an interesting research direction. For starters, there should be a much tighter analysis of our mixture of RSOL and Vickrey auctions. One avenue for improvement is to track contributions to the consumer surplus on the 83% of the probability space for which the chosen partition is not balanced. The “average balance” approach of Alaei et al. (2009), used previously to improve over the balanced partition technique of Feige et al. (2005) in a profit-maximization context, can be used to give such an improved bound. A second idea is to compare the mechanism’s consumer surplus directly to the benchmark in Theorem 3, rather than to the simpler “approximate benchmark” in Corollary 3. Analogously, avoiding the approximate benchmark  $\mathcal{F}^{(2)}$  could lead to better profit-maximizing platform designs (see Section V.B).

There are surely platforms that are universally adopted with smaller competitive advantage than is demonstrated by Theorem 5 in Section IV.C. A challenging problem is to characterize optimal platforms for the consumer surplus objective. Even the case of three-bidder single-item settings appears challenging. We conjecture that, for consumer surplus maximization with  $n$  bidders and a single item, the minimum competitive advantage required by an optimal platform for universal adoption is precisely the expected value of the performance benchmark when bidders’ valuations are drawn i.i.d. from the exponential distribution (as in Theorem 3).

While solving for optimal platforms is interesting theoretically, we suspect that optimal platforms will suffer from some drawbacks. First, when the number of agents is large, the optimal platform is a complex object, perhaps a distribution over a very large number of different auctions. This

complexity is characteristic of exact optimization in any auction analysis framework; other well-known examples include profit-maximizing auctions in Bayesian single-item settings when bidders' valuations are not identical or are i.i.d. from an irregular distribution (Myerson, 1981). The complexity of optimal auctions motivates the design and analysis of platforms that are relatively simple while requiring a competitive advantage that is almost as small as the minimum possible, e.g., in the spirit of Bulow and Klemperer (1996) and Hartline and Roughgarden (2009). Second, in optimizing a min-max criterion, the optimal platform will equalize the approximation factor of the benchmark across all valuation profiles (cf., the proof of Lemma 3). In practice there might be agreed-upon common distributions and rare distributions with auction performance on common distributions being the most important. For example, the random-sampling-based auctions of Balcan et al. (2008) out perform the optimal platform mechanism (Goldberg et al., 2006) on a natural family of common distributions.

An auction that approximates the performance benchmark is simultaneously near-optimal in every Bayesian setting with i.i.d. bidder valuations. The converse need not hold, and an interesting research direction is to better understand the relationship between these two conditions. Sometimes, as with the monopoly pricing problem studied in Section I, simultaneous Bayesian near-optimality is as hard as approximation of the performance benchmark.<sup>8</sup> This is not always the case, however. For example, Dhangwatnotai et al. (2014) showed that the digital good auction that partitions the agents into pairs and runs a Vickrey auction to serve one agent in each pair

<sup>8</sup>For a value of  $h \geq 1$ , consider the set of distributions that are concentrated at a single point in  $[1, h]$ . For each such distribution, the corresponding optimal auction extracts full surplus. As in Section I, no single auction can extract more than a  $1/\ln h$  fraction of the surplus for every such distribution.

obtains a 2-approximation to the revenue of the Bayesian optimal mechanism whenever the distribution is *regular*, meaning that virtual values are increasing (cf. Section II and Appendix D). The proof of this 2-approximation is a simple consequence of the  $n = 1$  agent special case of the main theorem of Bulow and Klemperer (1996), i.e., that the 2-agent Vickrey auction obtains more revenue than monopoly pricing a single agent. As mentioned in Section V.A, no auction for a digital good achieves a 2-approximation of the benchmark  $\mathcal{G}^{(2)}$  (Goldberg et al., 2006). On the other hand, all work thus far on simultaneous Bayesian near-optimality that avoids the pointwise benchmark approach — termed “prior-independent guarantees” by Dhangwatnotai et al. (2014) — are confined to regular distributions (Dhangwatnotai et al., 2014; Devanur et al., 2011; Roughgarden et al., 2012). By contrast, our benchmark approximations directly imply prior-independent guarantees for most distributions (for profit maximization) and for all distributions (for consumer surplus maximization).

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## APPENDIX

### A. DISTRIBUTION CONSTRUCTION FROM VIRTUAL VALUES

Section II gives a formula for calculating an agent’s virtual value function from the distribution from which her value is drawn. For the consumer surplus objective, this formula is  $\vartheta(v) = \frac{1-F(v)}{f(v)}$ . This section reverses the

calculation and gives a constructive proof that every non-negative piecewise constant function arises as the virtual value (for utility) function for some distribution. This fact is alluded to in Section III and is used explicitly in the proof in Appendix B of Theorem 4 from Section IV.B.

First observe that the exponential distribution has constant virtual value equal to its mean. That is, the exponential distribution with mean  $\mu$  has rate  $1/\mu$ , cumulative distribution  $F_\mu(z) = 1 - e^{-z/\mu}$ , and virtual value  $\vartheta_\mu(z) = \mu$ .

Now consider a non-negative piecewise constant function  $\vartheta : [0, \infty) \rightarrow \mathbb{R}_+$ , where the boundaries of each interval are given by  $a_0 = 0, a_1, \dots$ , and where the value of the function on the interval  $[a_j, a_{j+1})$  is  $\vartheta_j$ . We construct the distribution  $F$  with virtual value function  $\vartheta(\cdot)$  inductively. The starting interval is given by the distribution function  $F(z) = F_{\vartheta_0}(z)$  for  $z \in [a_0, a_1]$ . With the first  $j - 1$  intervals defined by  $F(z)$  for  $z \in [a_0, a_j]$ , we define the distribution function for the  $j$ th interval by  $F(a_j + z) = F_{\vartheta_j}(F_{\vartheta_j}^{-1}(F(a_j)) + z)$  for  $a_j + z \in [a_j, a_{j+1}]$ . Intuitively, this construction does a horizontal shift of the distribution function of the exponential distribution with mean  $\mu_j$  so that its height matches the height of the constructed (so far) distribution function at  $a_j$ . This construction is illustrated in Figure A1.

## B. INADEQUACY OF STANDARD AUCTIONS

In this section we prove Theorem 4 from Section IV.B. The proof, in fact, demonstrates a lower bound on the competitive advantage for universal adoption of any standard auction that grows proportionally to  $\sqrt{\log n}$  with the number  $n$  of agents.

**THEOREM 4:** *For every  $\rho > 1$  there is a sufficiently large  $n$  such that, for an  $n$ -agent, 1-unit setting, no mixture over standard auctions is universally*

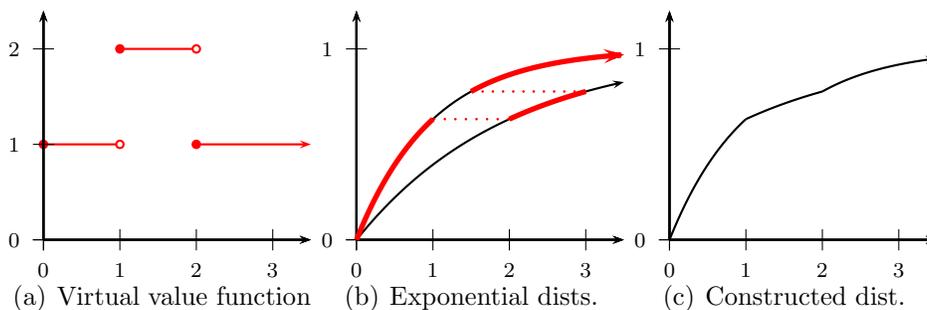


FIGURE A1. CONSTRUCTION OF THE CUMULATIVE DISTRIBUTION FUNCTION  $F$  WITH VIRTUAL VALUE FUNCTION (FOR UTILITY) THAT IS PIECEWISE CONSTANT ON  $[0, 1)$ ,  $[1, 2)$ , AND  $[2, \infty)$ , WITH VIRTUAL VALUES 1, 2, AND 1, RESPECTIVELY.

*adopted with competitive advantage  $\rho$ .*

Our argument uses the distributions in the construction in Appendix A; we next note some of their salient properties. These distributions are piecewise exponential distributions, with piecewise constant virtual values for utility and piecewise constant hazard rates. Recall that the virtual value for an exponential distribution equals its expected value which equals the reciprocal of its hazard rate. Also, exponential distributions are memoryless: given that the value  $v$  from an exponential distribution is at least  $z$ , the conditional distribution of  $v$  is identical to that of  $z + w$  where  $w$  is exponential with the same rate. Of particular relevance, the probability that an exponential random variable with mean one exceeds  $\beta$  is  $e^{-\beta}$ , and the probability that an exponential random variable with mean  $\beta$  exceeds  $\beta$  is  $1/e$ . For piecewise exponential distributions, these properties hold within each piece. From the analysis of Section II, the expected consumer surplus of any mechanism  $\mathcal{M}$  on distribution  $F$  is equal to its expected virtual surplus.

PROOF: Define  $\beta$  to be an integer greater than or equal to  $\max\{24 \cdot \rho, 2\}$ . For  $\kappa \in \{0, 1, 2, \dots, \beta - 1\}$ , let  $F_{\kappa, \beta}$  denote the piecewise exponential dis-

tribution (as in Appendix A) with virtual value for utility equal to 1 everywhere except on the interval  $[\kappa\beta, \kappa\beta + \beta)$ , where it is equal to  $\beta$ . Such a distribution thus has a “special interval” where the hazard rate is relatively low (and virtual value is relatively high). Let  $\mathcal{C}_\beta$  denote the set of distributions  $\{F_{\kappa,\beta}\}_{\kappa=0}^{\beta-1}$ . Consider a setting with a single item and  $n = e^{\beta^2}$  agents (rounded up to the nearest integer).

We claim the following:

- (a) For every  $F_{\kappa,\beta} \in \mathcal{C}_\beta$ , there is an auction with expected consumer surplus at least  $\beta/4$ .
- (b) For every standard auction  $\mathcal{A}$ , if  $F_{\kappa,\beta} \in \mathcal{C}_\beta$  is chosen uniformly at random, then the expected consumer surplus of  $\mathcal{A}$ , over the choice of  $F_{\kappa,\beta}$  and valuations  $v_1, \dots, v_n \sim F_{\kappa,\beta}$ , is at most 6.

Claims (a) and (b) imply the theorem. To see this, the consumer surplus of a convex combination of mechanisms  $\mathcal{M}$  is the convex combination of their consumer surpluses. As the inequality of property (b) holds for each auction in the support of such a convex combination, it also holds for the combination. Taking expectation over  $F$  uniform from  $\mathcal{C}_\beta$  in inequality (4) from Section III we have,

$$\mathbf{E}_{\mathbf{v},F}[\mathcal{G}(\mathbf{v})] \geq \mathbf{E}_{\mathbf{v},F}[\text{Opt}_F(\mathbf{v})] \geq \frac{\beta}{24}\mathbf{E}_{\mathbf{v},F}[\mathcal{M}(\mathbf{v})].$$

By the definition of expectation, there must exist a valuation profile  $\mathbf{v}$  that achieves this separation, i.e., with  $\mathcal{G}(\mathbf{v}) \geq \frac{\beta}{24}\mathcal{M}(\mathbf{v})$ . Thus, the competitive advantage needed for universal adoption is at least  $\beta/24 \geq \rho$ .

We now proceed to the proofs of (a) and (b). Call an agent *high-valued* if her value is at least  $\beta^2$ . The probability that there is no high-valued agent

is at most  $1/\beta$ .<sup>9</sup>

To prove (a), fix a choice of  $\kappa \in \{0, 1, \dots, \beta - 1\}$  and consider the  $\kappa\beta$ -lottery (Definition 4). The probability that all agents' values are below  $\kappa\beta$  is less than the probability that all agents' values are below  $\beta^2$ ; thus the probability of a winner in this lottery is at least  $1 - 1/\beta \geq 1/2$ . Otherwise, the winner is a random agent with value at least  $\kappa\beta$ . By the memoryless property of exponential distributions, the probability that the winner, with value at least  $\kappa\beta$ , has value less than  $\kappa\beta + \beta$  is  $1 - 1/e \geq 1/2$ ; such a winner has virtual value  $\beta$ . Virtual values for consumer surplus are non-negative, so the expected virtual surplus of the  $\kappa\beta$ -lottery is at least  $\frac{1}{2} \cdot \frac{1}{2} \cdot \beta$ , as claimed.

To prove (b), we first warm up by considering the case where  $\mathcal{A}$  is an  $r$ -lottery. Choose  $F_{\kappa,\beta} \in \mathcal{C}_\beta$  uniformly at random. The intuition is that this random choice effectively “hides” the location of the large virtual values.

If  $r \geq \kappa\beta + \beta$ , then the winner of  $\mathcal{A}$  (if any) has virtual value 1. If  $r \leq \kappa\beta - \beta$ , then by the memoryless property of exponential distributions, the value of a winner is less than  $\kappa\beta$  with probability at least  $1 - e^{-\beta}$ . Thus, the expected virtual value of a winner in this case (if any) is at most  $1 + \beta e^{-\beta} \leq 2$ . Finally, if  $r \in (\kappa\beta - \beta, \kappa\beta + \beta)$ , then the virtual value of a winner (if any) is at most  $\beta$ . As the third case occurs with probability at most  $2/\beta$  (over the random choice of  $\kappa$ ), the expected virtual value of  $\mathcal{A}$  is at most  $\frac{2}{\beta} \cdot \beta + 1 \cdot 2 = 4$ .

We conclude by extending the argument of the preceding paragraph for one-level lotteries to an arbitrary ironed virtual surplus maximizer  $\mathcal{A}$ . In our

<sup>9</sup>The analysis is elementary. Using the memoryless property of exponential distributions, the probability that a given agent is high-valued is  $\eta = (e^{-\beta})^{\beta-1} \cdot e^{-1} = e^{-\beta^2 + (\beta-1)}$ . With  $n = e^{\beta^2} = e^{\beta-1}/\eta$  agents, the probability of no high-valued agents is  $(1 - \eta)^n \leq e^{-\eta n} \leq e^{-e^{\beta-1}} \leq 1/\beta$ . The last inequality can be verified by checking the lower endpoint of  $\beta = 2$  and comparing the derivatives of  $e^{e^{\beta-1}}$  and  $\beta$ .

single-item symmetric setting, the auction  $\mathcal{A}$  specifies ironed intervals where ties are broken randomly but otherwise awards the item to the agent with the highest value. Let  $[r, r']$  denote the ironed interval of  $\mathcal{A}$  that contains the value  $\beta^2$ . (If value  $\beta^2$  is not ironed, then set  $r = r' = \beta^2$ .) Valuation profiles without a high-valued bidder occur with probability at most  $1/\beta$  (by the above analysis) and give virtual surplus at most  $\beta$ ; thus their contribution to the expected virtual surplus of  $\mathcal{A}$  is at most 1. Valuation profiles with a high-valued bidder and highest value in  $[r, r']$  contribute the same expected virtual surplus to  $\mathcal{A}$  as to a  $r$ -lottery; by the previous paragraph, this contribution is at most 4. In every valuation profile with highest value greater than  $r'$ , the item is awarded to a bidder with virtual value 1; these profiles contribute at most 1 to the expected virtual surplus of  $\mathcal{A}$ . Q.E.D.

### C. ANALYSIS OF RANDOM SAMPLING OPTIMAL LOTTERY MECHANISM

In this section we analyze the Random Sampling Optimal Lottery Mechanism (RSOL, Definition 5); which partitions to a sample and market, finds the optimal one-level lottery on the sample, and then runs this lottery on the market; and prove that for the consumer surplus objective there is a platform that is universally adopted with a finite competitive advantage. As described in Section IV.C, there are two steps to this proof. The first is showing that one-level lotteries guarantee at least half the consumer surplus of two-level lotteries. The second step shows that either the  $k$ -unit Vickrey auction or RSOL obtains nearly the consumer surplus of the optimal one-level lottery on the full valuation profile.

LEMMA 6: *For every valuation profile  $\mathbf{v}$  and parameters  $k, p$ , and  $q$ , there*

is an  $r$  such that the  $k$ -unit  $r$ -lottery obtains at least half of the expected consumer surplus of the  $k$ -unit  $(p, q)$ -lottery.

PROOF: We prove the lemma by showing that  $\text{Lot}_{p,q}(\mathbf{v}) \leq \text{Lot}_p(\mathbf{v}) + \text{Lot}_q(\mathbf{v})$ . We argue the stronger statement that each agent enjoys at least as large a combined expected utility in  $\text{Lot}_p(\mathbf{v})$  and  $\text{Lot}_q(\mathbf{v})$  as in  $\text{Lot}_{p,q}(\mathbf{v})$ .

Let  $S$  and  $T$  denote the agents with values in the ranges  $(p, \infty)$  and  $(q, p]$ , respectively. Let  $s = |S|$  and  $t = |T|$ . Assume that  $0 < s \leq k < s + t$  as otherwise the  $k$ -unit  $(p, q)$  lottery is equivalent to a one-level lottery. Each agent in  $T$  participates in a  $k$ -unit  $q$ -lottery in  $\text{Lot}_q$  and only a  $(k - s)$ -unit  $q$ -lottery in  $\text{Lot}_{p,q}$ ; her expected utility can only be smaller in the second case. Now consider  $i \in S$ . Writing  $\rho = (k - s + 1)/(t + 1)$  in equation (5) we can upper bound the utility of agent  $i$  in  $\text{Lot}_{p,q}$  by

$$v_i - p + \rho(p - q) = (1 - \rho)(v_i - p) + \rho(v_i - q) \leq (v_i - p) + \frac{k}{s+t} \cdot (v_i - q),$$

which is the combined expected utility that the agent obtains from participating in both a  $k$ -unit  $p$ -lottery (with  $s \leq k$ ) and a  $k$ -unit  $q$ -lottery. Q.E.D.

COROLLARY 3: *For every valuation profile  $\mathbf{v}$ , the supremum of Bayesian optimal mechanisms benchmark  $\mathcal{G}$  is at most twice the expected consumer surplus of the best one-level lottery:  $\mathcal{G}(\mathbf{v}) \leq 2 \cdot \sup_r \text{Lot}_r(\mathbf{v})$ .*

A key fact that enables the analysis of RSOL is that, with constant probability, the relevant statistical properties of the full valuation profile are preserved in the market and the sample. These statistical properties can be summarized in terms of a “balance” condition. Define a partition of the agents  $\{1, 2, 3, \dots, n\}$  into a market  $M$  and a sample  $S$  to be *balanced* if

$1 \in M$ ,  $2 \in S$ , and for all  $i \in \{3, \dots, n\}$ , between  $i/4$  and  $3i/4$  of the  $i$  highest-valued agents are in  $S$  (and similarly  $M$ ). In the proof of Theorem 5, we use the following adaptation of the “Balanced Sampling Lemma” of Feige et al. (2005) to bound from below the probability that RSOL selects a balanced partitioning (for completeness we include this lemma’s proof below).

LEMMA 7: *When each agent is assigned to the market  $M$  or sample  $S$  independently according to a fair coin, the resulting partitioning is balanced with probability at least 0.169.*

PROOF: Call a subset of the agents *imbalanced* if, for some  $i \geq 3$ , it contains fewer than  $i/4$  of the  $i$  highest-valued agents. After conditioning on the events that  $1 \in M$  and  $2 \in S$ , the probability that  $S$  is imbalanced can be calculated as at most 0.161 by a simple probability of ruin analysis proposed in Feige et al. (2005) (details given below). By symmetry, the same bound holds for  $M$ . By the union bound, the partition is balanced with probability at least 0.678. Agent 1 is in  $M$  and 2 is in  $S$  with probability  $1/4$ , so the unconditional probability that the partition is balanced is at least 0.169.

The following analysis from Feige et al. (2005) shows that the conditional probability that  $S$  is imbalanced is at most 0.161. Consider the random variable  $Z_i = 4|S \cap \{1, \dots, i\}| - i$ ; the balanced condition is equivalent to  $Z_i \geq 0$  for all  $i \geq 3$ . By the conditioning,  $S \cap \{1, 2\} = \{2\}$  and so  $Z_2 = 2$ . View  $Z_i$  as the positions of a random walk on the integers that starts from position two and takes three steps forward (at step  $i$  with  $i \in S$ ) or one step back (at step  $i$  with  $i \notin S$ ), each with probability  $1/2$ . The set  $S$  is imbalanced if and only if this random walk visits position  $-1$ . The probability  $r$  of ever (with  $n \rightarrow \infty$ ) visiting the preceding position in such

a random walk can be calculated as the root of  $r^4 - 2r + 1$  on the interval  $(0, 1)$ , which is approximately 0.544. The probability of imbalance, which requires eventually moving backward three steps from position 2, is at most  $r^3 \leq 0.161$ , as claimed. Q.E.D.

**THEOREM 5:** *For every  $n, k \geq 1$ , there is an  $n$ -agent  $k$ -unit platform mechanism that is universally adopted with constant competitive advantage.*

**PROOF:** We outline the high-level argument and then fill in the details. We focus on the expected consumer surplus of RSOL, where the expectation is over the random partition of agents, relative to that of an optimal one-level lottery, on the “truncated” valuation profile  $\mathbf{v}^{(2)} = (v_{(2)}, v_{(2)}, v_{(3)}, \dots, v_{(n)})$ . We only track the contributions to RSOL’s expected consumer surplus when the partitioning of the agents is balanced. In such cases, RSOL’s consumer surplus on the truncated valuation profile can only be less than on the original one.

Step 1 of the analysis proves that, conditioned on the partitioning of the agents being balanced, the expected consumer surplus of the optimal one-level lottery for the sample is at least  $1/2$  times that of the optimal one-level lottery for the full truncated valuation profile. Step 2 of the analysis proves that, conditioned on an arbitrary balanced partition, the consumer surplus of every one-level lottery on the market is at least  $1/9$  times its consumer surplus on the sample. In particular, this inequality holds for the optimal one-level lottery for the sample. Combining these two steps with Lemma 7 implies that the expected consumer surplus of RSOL is at least  $0.169 \times \frac{1}{2} \times \frac{1}{9} \geq 1/107$  times that of the optimal one-level lottery on the truncated valuation profile  $\mathbf{v}^{(2)}$ . The additional consumer surplus achieved by an optimal one-level lottery on the original valuation profile  $\mathbf{v}$

over the truncated one is at most  $v_{(1)} - v_{(2)}$ . The consumer surplus of the  $(k + 1)$ th-price auction, where  $k$  is the number of units for sale, is at least this amount. The platform mechanism that mixes between RSOL with probability  $107/108$  and the  $(k + 1)$ th-price auction with probability  $1/108$  has expected consumer surplus at least  $1/108$  times that of the optimal one-level lottery on  $\mathbf{v}$ , and (by Corollary 3) at least  $1/216$  times the benchmark  $\mathcal{G}$ . Below, we elaborate on the two steps described above.

Step 1: *Conditioned on a balanced partitioning, the expected consumer surplus of the optimal one-level lottery for the sample  $S$  is at least  $1/2$  times that of the optimal one-level lottery for the full truncated valuation profile.* Let  $r$  be the price of the optimal one-level lottery for  $\mathbf{v}^{(2)}$ . Conditioned on a balanced partition, exactly one of the top two (equal-valued) bidders of  $\mathbf{v}^{(2)}$  lies in  $S$ . By symmetry, each other bidder has probability  $1/2$  of lying in  $S$ . The winning probability of bidders in  $S$  with value at least  $r$  is only higher than that when all agents are present. Summing over the bidders' contributions to the consumer surplus and using the linearity of expectation,  $\mathbf{E}_S[\text{Lot}_r(S) \mid \text{balanced partition}] \geq \text{Lot}_r(\mathbf{v}^{(2)})/2$ . Of course, the optimal one-level lottery for the sample is only better.

Step 2: *Conditioned on an arbitrary balanced partition, for the truncated valuation profile  $\mathbf{v}^{(2)}$ , the consumer surplus of every one-level lottery on the market is at least  $1/9$  times its consumer surplus on the sample.* Fix a balanced partition into  $S$  and  $M$  and a one-level lottery at price  $r$ . The expected contribution of a bidder  $j$  to a  $r$ -lottery is  $(v_j - r)$  times its winning probability (if  $v_j > r$ ) or 0 (otherwise). The balance condition ensures that, for every  $i \geq 2$ , the number of the  $i$  highest-valued bidders that belong to the market is between  $1/3$  and 3 times that of the sample. In particular,

the winning probability of bidders with value at least  $r$  in  $M$  is at least  $1/3$  of that of such bidders in  $S$ . Moreover, the balance condition implies that

$$\sum_{j \in M} \max\{v_j - r, 0\} \geq \frac{1}{3} \sum_{j \in S} \max\{v_j - r, 0\}$$

for the truncated valuation profile  $\mathbf{v}^{(2)}$ ; the claim follows. Q.E.D.

It is certainly possible to optimize better the parameters of the platform mechanism defined in the proof of Theorem 5. Furthermore, since for simplicity we only keep track of RSOL's performance when the partition is balanced, the mechanism's performance is better than the proved bound.

#### D. PROFIT MAXIMIZATION WITH NEAR-UNIVERSAL ADOPTION

Recall from Section V the benchmark  $\mathcal{G}^{(2)} = \max_{i \geq 2} iv_{(i)}$ , which effectively excludes selling to the highest-valued agent at her value. We will show that a mechanism  $\mathcal{M}$  that achieves a  $\beta$ -approximation of this benchmark on every valuation profile is near-universally adopted with competitive advantage  $\beta$ , meaning that for every distribution  $F$  in a large class, the expected profit of  $\mathcal{M}$  is at least a  $\beta$  fraction of that of the Bayesian optimal auction for  $F$ . By Proposition 1, it suffices to give sufficient condition on  $F$  that guarantees that  $\mathbf{E}_{\mathbf{v}}[\mathcal{G}^{(2)}(\mathbf{v})] \geq \mathbf{E}_{\mathbf{v}}[\text{Opt}_F(\mathbf{v})]$ .

From Bulow and Roberts (1989), virtual values for revenue are given by the *marginal revenue* of the *revenue curve* that plots the revenue  $p(1 - F(p))$  against the probability  $1 - F(p)$  that the agent buys (i.e., her expected demand). Virtual values are given by the slope of the revenue curve, thus monotonicity of virtual values, as required by the regularity condition of Myerson (1981), is equivalent to the concavity of the revenue curve.

Our sufficient condition, the “inscribed triangle property,” states that for

every point  $(1 - F(p), p(1 - F(p)))$  on the revenue curve, the triangle formed with the points  $(0, 0)$  and  $(1, 0)$  lies underneath the revenue curve. This condition is clearly satisfied whenever the revenue curve is concave — equivalently, whenever the distribution is regular — and is also satisfied by a large family of multi-modal distributions that are not regular.

To understand this condition better, observe that, for every distribution  $F$  and price  $p$ , the line from  $(0, 0)$  to  $(1 - F(p), p(1 - F(p)))$  lies beneath the revenue curve. The reason is that, for every  $\alpha \in [0, 1]$ , the price  $p'$  with selling probability  $\alpha \cdot (1 - F(p))$  is at least  $p$  and hence obtains revenue  $p'(1 - F(p')) \geq \alpha \cdot p(1 - F(p))$ . The inscribed triangle property is therefore equivalent to requiring that, for every price  $p$ , the line between  $(1 - F(p), p(1 - F(p)))$  and  $(1, 0)$  lies beneath the revenue curve. For an economic interpretation of this condition, consider the measure of types that are not served at a price  $p$ , i.e.,  $F(p)$ . Viewing the revenue curve as a function of  $F(p)$ , the condition says that as the price is dropped, the revenue per unit of types that are not served is non-decreasing. In other words, the condition requires  $p(1 - F(p))/F(p)$  to be non-increasing.

The inscribed triangle property immediately implies the following lemma, which is reminiscent of the main theorem of Bulow and Klemperer (1996).

LEMMA 8: *For distribution  $F$  with non-increasing  $v(1 - F(v))/F(v)$ , the two-agent Vickrey auction revenue exceeds the single-agent optimal revenue.*

PROOF: In the two-agent Vickrey auction, each agent faces a take-it-or-leave-it offer equal to the other agent's bid. Thus, each agent faces a random price  $p$  distributed such that the probability of sale to this agent is uniform on  $[0, 1]$ . Since the revenue of every such price is given by the revenue curve, and the distribution of  $1 - F(p)$  is uniform, the expected revenue

obtained from the agent equals the area under the revenue curve. Invoking the inscribed triangle property at the point  $(1 - F(p^*), p^*(1 - F(p^*)))$  for the monopoly price  $p^*$ , we conclude that the expected revenue obtained from one agent is at least  $\frac{1}{2} \cdot 1 \cdot p^*(1 - F(p^*))$ , half the expected revenue of the monopoly price. Since there are two agents, the total expected Vickrey revenue is at least the optimal single-agent revenue. Q.E.D.

LEMMA 9: *For distribution  $F$  with the non-increasing  $v(1 - F(v))/F(v)$  property, conditioning to exceed a price  $p$  preserves the property.*

PROOF: The original condition is saying that the virtual value is not more negative than the slope of the line that connects that point on the revenue curve to  $(1, 0)$ . After conditioning on being at least  $p$ , the condition can be viewed on the original revenue curve as the virtual value not being more negative than the slope of line that connects the point on the revenue curve to  $(1 - F(p), 0)$ . As this slope is steeper than the slope of the line through  $(1, 0)$ , the property is preserved by such conditioning. Q.E.D.

We now combine the above lemmas to prove Lemma 5, restated below. A key intuition in this proof is that Lemma 8 implies that  $\mathbf{E}_{\mathbf{v}}[2v_{(2)}] \geq \mathbf{E}_{\mathbf{v}}[\text{Opt}_F(\mathbf{v})]$  for  $k = 2$  items and  $n = 2$  agents — the left-hand side is double the revenue of the Vickrey auction (with 1 item) and the right-hand side is double the revenue of the single-agent optimal mechanism.

LEMMA 10: *For digital good settings and every distribution  $F$  with  $v(1 - F(v))/F(v)$  non-increasing,  $\mathbf{E}_{\mathbf{v}}[\mathcal{G}^{(2)}(\mathbf{v})] \geq \mathbf{E}_{\mathbf{v}}[\text{Opt}_F(\mathbf{v})]$ .*

PROOF: Let  $p^* = \text{argmax}_p p(1 - F(p))$  be the monopoly price for the distribution. The analysis proceeds by conditioning on  $v_{(3)} = z$  and considering

the cases where  $z \leq p^*$  and  $z > p^*$ . In the first case, we have

$$\begin{aligned} \mathbf{E}_{\mathbf{v}}[\mathcal{G}^{(2)}(\mathbf{v}) \mid v_{(3)} = z \leq p^*] &\geq \mathbf{E}_{\mathbf{v}}[2v_{(2)} \mid v_{(3)} = z \leq p^*] \\ &\geq \mathbf{E}_{\mathbf{v}}[\text{Opt}_F(\mathbf{v}) \mid v_{(3)} = z \leq p^*]. \end{aligned}$$

The last inequality follows by Lemmas 8 and 9 and the fact that, given  $v_{(3)} = z \leq p^*$ ,  $\text{Opt}_F$  is an auction that sells to at most agents 1 and 2 and therefore has revenue that is at most the optimal auction that sells to these agents for the conditional distribution. In the second case, let  $k^* \geq 3$  be a random variable for the number of units sold by  $\text{Opt}_F$ ; we have

$$\begin{aligned} \mathbf{E}_{\mathbf{v}}[\mathcal{G}^{(2)}(\mathbf{v}) \mid v_{(3)} = z \geq p^*] &\geq \mathbf{E}_{\mathbf{v}}[k^* v_{(k^*)} \mid v_{(3)} = z \geq p^*] \\ &\geq \mathbf{E}_{\mathbf{v}}[k^* p^* \mid v_{(3)} = z \geq p^*] \\ &= \mathbf{E}_{\mathbf{v}}[\text{Opt}_F(\mathbf{v}) \mid v_{(3)} = z \geq p^*]. \end{aligned}$$

Combining the two cases proves the lemma. Q.E.D.