MATHEMATICS OF OPERATIONS RESEARCH
Vol. 00, No. 0, Xxxxx 0000, pp. 000–000
ISSN 0364-765X | EISSN 1526-5471 | 00 | 0000 | 0001

INFORMS
DOI 10.1287/xxxx.0000.0000


# Restoring Pure Equilibria to Weighted Congestion Games*

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Congestion games model several interesting applications, including routing and network formation games, and also possess attractive theoretical properties, including the existence of and convergence of natural dynamics to a pure Nash equilibrium. Weighted variants of congestion games that rely on sharing costs proportional to players' weights do not generally have pure-strategy Nash equilibria. We propose a new way of assigning costs to players with weights in congestion games that recovers the important properties of the unweighted model. This method is derived from the Shapley value, and it always induces a game with a (weighted) potential function. For the special cases of weighted network cost-sharing and weighted routing games with Shapley value-based cost shares, we prove tight bounds on the price of stability and price of anarchy, respectively.

*Key words*: price of anarchy; price of stability; Shapley value; congestion games
*MSC2000 subject classification*: Primary: 91A10; secondary: 91A43, 90C90
*OR/MS subject classification*: Primary: games, noncooperative; secondary: networks, multicommodity

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**1. Introduction.** Congestion games are a well-studied generalization of several game-theoretic models, including some fundamental network formation games and routing games. In the standard model [22], there is a ground set of resources, and each player has a set of allowable strategies, each of which is a subset of resources. For example, the strategies of a player could correspond to the paths of a network with a given source and sink. Each resource has a per-user cost that depends on the number of players that use it, and the goal of each player is to minimize the sum of the resources' costs in its strategy, given the strategies chosen by the other players. In atomic selfish routing games [23, 26], strategies correspond to paths and the per-unit cost function $c_e(\cdot)$ of each resource $e$ is nondecreasing. In network cost-sharing games [2], strategies correspond to paths and the (decreasing) cost functions have the form $c_e(x_e) = \gamma_e/x_e$, where $\gamma_e$ is the fixed installation cost of edge $e$ and $x_e$ is the number of players that share it.

A *pure Nash equilibrium (PNE)* is a strategy profile such that no player can decrease its cost via a unilateral deviation. Many games, such as "Rock-Paper-Scissors", have no PNE. Rosenthal [22] used a potential function argument to show that every congestion game — and thus every atomic selfish routing and network cost-sharing game — has at least one PNE. Moreover, best-response dynamics is guaranteed to converge to a PNE [18].

* A preliminary version of this paper appeared in the Proceedings of the 38th Annual Colloquium on Automata, Languages and Programming, July 2011.

† Supported in part by an ONR PECASE Award of the second author.

‡ Supported in part by NSF grants CCF-1016885 and CCF-1215965, an ONR PECASE Award, and an AFOSR MURI grant.

Every player of a congestion game imposes the same load on a resource. There are many motivations for relaxing this assumption and allowing non-uniform resource consumption. For example, in a network context, players could have different durations of resource usage, different bandwidth requirements, or different contracts with the network provider. Almost all research to date has modeled non-uniform players in congestion-type games through *proportional cost sharing* [1, 2, 3, 4, 5, 8, 11, 12, 17, 18]. The first assumption in proportional cost sharing is that each player $i$ has a positive weight $w_i$, with larger weights indicating larger resource usage. To explain the second assumption in a general way, let $C_e(S_e)$ denote the joint cost incurred by the subset $S_e$ of users of the resource $e$. For example, in a network cost-sharing game, $C_e(S_e)$ is the fixed cost $\gamma_e$ provided $S_e$ is non-empty (and is 0 otherwise). In (weighted) atomic selfish routing, $C_e(S_e)$ is $x_e \cdot c_e(x_e)$, where $c_e(\cdot)$ is the per-flow unit resource cost function and $x_e = \sum_{i \in S_e} w_i$ is the total weight of the players using $e$. Proportional cost sharing dictates that each player $i \in S_e$ pays a $w_i / \sum_{j \in S_e} w_j$ fraction of $C_e(S_e)$ for the resource $e$.

Unfortunately, most of the attractive theoretical properties of congestion games do not carry over to their weighted counterparts with proportional cost sharing. Network cost-sharing games with at least three players need not have a PNE [5]. Even when PNE do exist in such games, they can be much costlier (relative to an optimal solution) than in the unweighted case [2, 5]. Atomic selfish routing games with weighted players do not generally have PNE [9, 11, 23], except when all cost functions are affine [7] and in some other isolated special cases [11].

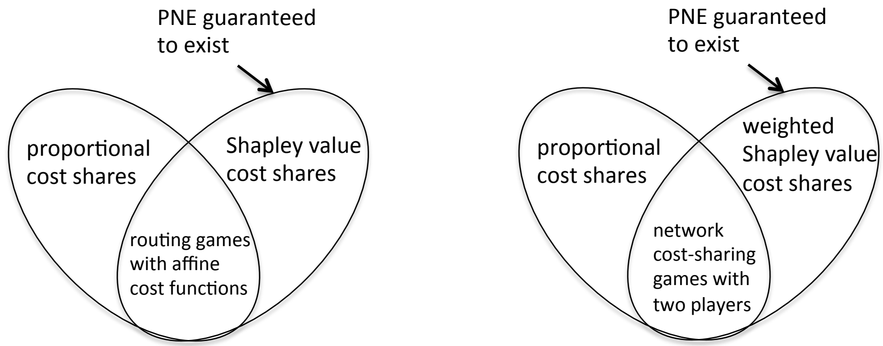
Guaranteed existence of PNE is an important property. There are, of course, more general equilibrium concepts like the mixed-strategy Nash equilibrium that are guaranteed to exist in every finite game, but randomized solution concepts suffer from well-known drawbacks in interpretation and implementation (see e.g. [21, §3.2]). Particularly when designing or influencing the game being played, there is good reason to make design decisions that guarantee the existence of and convergence of natural dynamics to a PNE. Previous works have studied how to design systems with such guarantees in the domains of queuing [19, 20, 28], network cost-sharing [6, 10], and distributed resource allocation [16].

**1.1. Our Contributions.** We propose a new way of assigning costs to players with weights in congestion-type games, which is derived from the Shapley value. We call the resulting class of games *SV weighted congestion games*. Extending work of Hart and Mas-Colell [13], we show that every SV weighted congestion game admits a (weighted) potential function. The existence of and convergence of natural dynamics to a PNE in every such game follow immediately.

For example, for the special case of atomic selfish routing games, we derive the cost shares for the users $S_e$ of edge $e$ by applying the standard Shapley value (defined in the next section) to the cost function $C_e(\cdot)$ above with the player set $S_e$. Since the incremental effect of a player on the joint cost is increasing in its weight, so is its cost share. These Shapley value-based cost shares coincide with proportional shares when all per-user cost functions are affine, but not otherwise (Figure 1(a)). These results explain the previously mysterious fact that the traditional proportional cost shares always yield a potential game if and only if all cost functions are affine [7, 11].

For the special case of network cost-sharing games, the symmetric joint cost function $C_e(\cdot)$ is insensitive to players' weights. To introduce weight-dependent cost shares, we use the *weighted* Shapley value [14, 27], which averages over orderings of the players in a non-uniform way (see the next section for a definition). The resulting cost shares are increasing in weight, and coincide with proportional shares (for all weight vectors) if and only if there are at most two players (Figure 1(b)). These facts explain why, with proportional cost shares, PNE always exist with two players [2] but not with at least three [5].

We also provide tight bounds on the inefficiency of equilibria in SV weighted network cost-sharing and atomic selfish routing games. For weighted atomic selfish routing games, we give tight bounds




(a)Atomic Routing Games (b)Network Cost-Sharing Games

FIGURE 1. Comparison of traditional proportional cost shares with the Shapley value cost shares proposed in the present work.

TABLE 1. The POA in weighted routing games with polynomial cost functions with nonnegative coefficients, for proportional cost shares (when PNE exist) and for Shapley cost shares.

| Degree | Proportional | Shapley |
|---|---|---|
| 1 | 2.618 | 2.618 |
| 2 | 9.909 | 12.626 |
| 3 | 47.82 | 101.58 |
| 4 | 277.0 | 1,117.78 |
| 5 | 1,858 | 15,195 |
| 6 | 14,099 | 244,399 |
| 7 | 118,926 | 4,536,010 |
| 8 | 1,101,126 | 95,410,300 |
| d | $\Theta(\frac{d}{\log d})^{d+1}$ | $\Theta(d)^{d+1}$ |

on the worst-case price of anarchy (POA) [15] — the ratio between the cost of the worst PNE and of an optimal outcome — with respect to every set of convex cost functions and a worst-case set of player weights. This worst-case POA is slightly larger than that in weighted congestion games with proportional cost shares that have PNE. For example, in routing games with cost functions that are polynomials with degree at most $d$ and nonnegative coefficients, the POA with proportional cost shares is $\approx (c_1 d/\ln d)^{d+1}$ (when PNE exist) [1] and with Shapley value cost shares is $\approx (c_2 d)^{d+1}$, where $c_1 \approx 1.3$ and $c_2 \approx 0.9$ are constants independent of $d$. See also Table 1. We establish these POA upper bounds with a "smoothness proof" in the sense of Roughgarden [24], so these upper bounds apply more generally to all mixed Nash, correlated, and coarse correlated equilibria of these games. Thus, Shapley cost shares restore PNE to weighted routing games at the expense of modestly more inefficiency.

For network cost-sharing games, we focus on the price of stability (POS) [2], which is the ratio between the cost of the best PNE and of an optimal solution. The worst-case POA is uninteresting in such games because it equals $k$, the number of players, no matter how players' cost shares are defined [6, Proposition 4.12]. Our main result here is a characterization of the POS as a function of the weight vector $w$. For every $w$, we give an explicit lower bound on the POS and prove a matching upper bound for all networks. The special case of $w = (1, 1, \ldots, 1)$ — where the worst-case POS is the $k$th Harmonic number — is one of the main results in Anshelevich et al. [2]. Our lower bound is a straightforward extension of that in [2], but our matching upper bound requires a fundamentally new argument. The upper bound in [2] for unweighted players follows directly from the proximity between the potential and objective functions; with weighted players, the difference between these two functions can be arbitrarily larger than the POS. Our characterization implies, for example, that the POS remains $O(\log k)$ if players' weights differ by a constant factor, and is $O(\sqrt{k})$ when $w_i = i$ for $i = 1, 2, \ldots, k$. With proportional cost shares, when PNE exist, the POS can be as large as the sum of the players' weights (assuming that $\min_i w_i = 1$) [5]. In this sense, weighted Shapley cost shares both restore PNE to weighted network cost-sharing games and decrease the inefficiency of such equilibria.

## 2. The Weighted Shapley Value.

We first recall the weighted Shapley value [14, 27]. Consider a set $S$ of players and a cost function $C : 2^S \to \mathbb{R}$. (For us, $S$ is the users of a resource and $C(T)$ is the joint cost that would be incurred if the players of $T \subseteq S$ were its sole users.) For a given ordering $\pi$ of the players, let $\Delta_i(\pi)$ denote $C(S^i(\pi) \cup \{i\}) - C(S^i(\pi))$, where $S^i(\pi)$ denotes the players preceding $i$ in $\pi$.

Each player has a positive weight $w_i$ and a sampling parameter $\lambda_i$ set to $1/w_i$ [14, 27]. We use the $\lambda_i$'s to define a distribution over orderings of players, as follows. (When all $\lambda_i$'s are equal, we recover the uniform distribution and the usual Shapley value.) We first choose the final player in the ordering, with probabilities proportional to the $\lambda_i$'s; given this choice, we choose the penultimate player randomly from the remaining ones, again with probabilities proportional to the $\lambda_i$'s; and so on. The weighted Shapley value of player $i$ is defined as the expected value of $\Delta_i(\pi)$ with respect to this distribution over orderings $\pi$.

## 3. Congestion Games with Shapley Value Cost Shares.

Sections 3.1 and 3.2 propose novel cost shares with weighted players in network cost-sharing games and routing games, respectively, which ensure the existence of pure-strategy Nash equilibria. Section 3.3 explains the general construction for arbitrary congestion games.

### 3.1. Network Cost-Sharing Games.

In an *SV network cost-sharing game*, each player $i = 1, 2, \ldots, k$ has a weight $w_i \geq 1$ and a sampling parameter $\lambda_i = 1/w_i$. We can assume that $w_1 \leq w_2 \leq \ldots \leq w_k$ and we do so for the rest of the paper. Player $i$ aims to construct a path $P_i$ from a given node $s_i$ to a given node $t_i$ in a directed graph $G = (V, E)$, where every $e \in E$ has a fixed nonnegative cost $\gamma_e$. With respect to a fixed path vector $P$, we write $S_e$ for the users of edge $e$. The cost function $C_e$ corresponding to edge $e$ is $C_e(S_e) = \gamma_e$ if $S_e \neq \emptyset$ and $C_e(S_e) = 0$ if $S_e = \emptyset$.

We next give a probabilistic representation of weighted Shapley cost shares and the corresponding potential function, in terms of independent exponentially distributed random variables. Let $T$ be a subset of the players. For every player $i \in T$, let $\mathbf{X}_i$ be an exponentially distributed random variable with rate $\lambda_i$. We then define the per-unit weighted Shapley share of $i$ on an edge $e$ used by the players $T$ as the probability that $X_i$ is the largest random variable among those associated with $T$.

Definition 3.1. In an SV network cost-sharing game, the weighted Shapley share of player $i \in S_e$ for using the edge $e$ is

$$\chi_{i,e}(S_e) = \gamma_e \cdot \mathbf{Pr}\left[\mathbf{X}_i = \max_{j \in S_e} \mathbf{X}_j\right]. \tag{1}$$

For the joint cost functions under discussion (equal to $\gamma_e$ for every non-empty set), Definition 3.1 coincides with the definition given in Section 2. Precisely, the distribution over orderings $\pi$ described in Section 2 is the same as the distribution induced by the relative values of exponential random variables with rates $\lambda_i$, sorted from largest to smallest [14]. Since the value $\Delta_i(\pi)$ is $\gamma_e$ for the first player of $\pi$ and 0 otherwise, the equivalence follows.

Weighted Shapley shares are always increasing in a player's weight. If a set $S_e$ contains at most two players, then the cost shares of Definition 3.1 are proportional to the players' weights. This is not generally true with three or more players.

EXAMPLE 3.1. Suppose $\gamma_e = 1$ and $S_e = \{1, 2\}$ with $w_1 = 1$ and $w_2 = 2$. Since the edge has unit cost, the weighted Shapley share of player 1 is the probability that 1 is first in the random ordering described in Section 2. Hence it is equal to the probability that 2 is picked in the first sampling step, which gives us

$$\chi_{1,e}(\{1,2\}) = \frac{\frac{1}{w_2}}{\frac{1}{w_1} + \frac{1}{w_2}} = \frac{1}{3}.$$

Similarly, $\chi_{2,e}(\{1,2\}) = 2/3$, and the cost shares are proportional to the players' weights. Now suppose that player 3 with $w_3 = 1$ joins edge $e$. The weighted Shapley share of 1 is again the probability that 1 is first in the random ordering. This is now

$$\chi_{1,e}(\{1,2,3\}) = \frac{\frac{1}{w_2}}{\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}} \cdot \frac{\frac{1}{w_3}}{\frac{1}{w_1} + \frac{1}{w_3}} + \frac{\frac{1}{w_3}}{\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}} \cdot \frac{\frac{1}{w_2}}{\frac{1}{w_1} + \frac{1}{w_2}} = \frac{7}{30}.$$

Since $w_3 = w_1$, we also have $\chi_{3,e}(\{1,2,3\}) = 7/30$, and then $\chi_{2,e}(\{1,2,3\}) = 1 - 2 \cdot \frac{7}{30} = 8/15$. These cost shares are not proportional to the players' weights.

We next show that every SV network cost-sharing game with the cost shares of Definition 3.1 admits a (weighted) potential function. Define the function $\Phi$ by

$$\Phi(P) = \sum_{e \in E} \Phi_e(P), \tag{2}$$

where the edge potential $\Phi_e$ is defined as

$$\Phi_e(P) = \gamma_e \cdot \mathbf{E}\left[\max_{j \in S_e} \mathbf{X}_j\right].$$

PROPOSITION 3.1. *For every pair $P$ and $P' = (P_{-i}, P'_i)$ of path vectors that differ only in the $i$th component,*

$$\Phi(P') - \Phi(P) = w_i \cdot (C_i(P') - C_i(P)), \tag{3}$$

*where $C_i$ denotes the sum of the cost shares paid by player $i$.*

*Proof.* We prove that every edge contributes the same amount to the left- and right-hand sides of (3). If $e \in P_i \cap P'_i$ or $e \notin P_i \cup P'_i$, there is nothing to prove. By symmetry, we can assume that $e \in P'_i \setminus P_i$. We need to show that

$$\Phi_e(P') - \Phi_e(P) = w_i \cdot \chi_{i,e}(S_e \cup \{i\}), \tag{4}$$

where $S_e$ is the set of players that use $e$ in $P$.

The left-hand side of (4) is the difference between

$$\Phi_e(P') = \gamma_e \cdot \mathbf{E}\left[\max_{j \in S_e \cup \{i\}} \mathbf{X}_j\right] \quad \text{and} \quad \Phi_e(P) = \gamma_e \cdot \mathbf{E}\left[\max_{j \in S_e} \mathbf{X}_j\right].$$

The maxima inside the expectations are different only when $\mathbf{X}_i$ is larger than the corresponding random variable of every player of $S_e$. Conditioning on this event and using the fact that the exponential distribution is memoryless, the conditional expected difference between the two maxima is $1/\lambda_i = w_i$. Hence $\Phi_e(P') - \Phi_e(P) = w_i \cdot \gamma_e \cdot \mathbf{Pr}[\mathbf{X}_i = \max_{j \in T} \mathbf{X}_j] = w_i \cdot \chi_{i,e}(S_e \cup \{i\})$, as claimed. □

As in Rosenthal [22] and Monderer and Shapley [18], the existence of a weighted potential function has immediate consequences. First, by (3), the outcome with minimum potential function value is a PNE. Moreover, every iteration of best-response dynamics — in which a player switches strategies to strictly decrease its cost — strictly decreases the potential function. Thus, best-response dynamics converges, necessarily to a PNE.

COROLLARY 3.1. *In every SV network cost-sharing game, best-response dynamics converges to a PNE.*

**3.2. SV Atomic Selfish Routing.** In a *SV atomic selfish routing game*, each player $i = 1, 2, \ldots, k$ has a weight $w_i$ and selects a path $P_i$ from a node $s_i$ to a node $t_i$ in a given graph $G = (V, E)$. For every edge $e \in E$, the per-unit cost function $c_e(\cdot)$ is nonnegative and nondecreasing. Its users $S_e$ have to pay a joint cost of

$$C_e(S_e) = x_e \cdot c_e(x_e), \tag{5}$$

where $x_e$ is their total weight.

The joint cost function (5) is asymmetric, meaning that its value depends on the identities of the players in the set $S_e$ and not just on $|S_e|$. This in is a contrast with weighted network cost-sharing games, where the asymmetry was exogenous to the (symmetric) joint cost function. For this reason, the standard (unweighted) Shapley value already gives meaningful weight-dependent cost shares in routing game with non-uniform player weights, and these are the cost shares proposed below. That is, we take the sampling parameter $\lambda_i$ from Section 2 to be 1 for every player $i$ (and not $1/w_i$). Section 3.3 outlines a natural generalization that accommodates both asymmetric cost functions and exogenous player asymmetry.

DEFINITION 3.2. In an SV atomic selfish routing game, the Shapley share of player $i \in S_e$ on edge $e$ is

$$\chi_{i,e}(S_e) = \mathbf{E}\left[C_e(S_e^i(\pi_e) \cup \{i\}) - C_e(S_e^i(\pi_e))\right],$$

where $S_e^i(\pi_e)$ denotes the players preceding $i$ in $\pi_e$, a uniformly random ordering of $S_e$.

The cost shares in Definition 3.2 are generally proportional to players' weights if and only if the per-unit cost function $c_e$ is affine.

EXAMPLE 3.2. Suppose $c_e(x) = x$ and $S_e = \{1, 2\}$ with $w_1 = 1$ and $w_2 = 2$. Then the joint cost that the players have to share is $(w_1 + w_2)^2 = 9$. The Shapley share of player 1 is

$$\chi_{1,e}(\{1,2\}) = \frac{1}{2} \cdot w_1^2 + \frac{1}{2} \cdot \left((w_1 + w_2)^2 - w_2^2\right) = 3.$$

Similarly we get $\chi_{2,e} = 6$ and see that the cost shares are proportional. Now suppose that $c_e(x) = x^2$ and $S_e$ remains the same. The joint cost is $(w_1 + w_2)^3 = 27$ and

$$\chi_{1,e}(\{1,2\}) = \frac{1}{2} \cdot w_1^3 + \frac{1}{2} \cdot \left((w_1 + w_2)^3 - w_2^3\right) = 10, \quad \text{and}$$
$$\chi_{2,e}(\{1,2\}) = \frac{1}{2} \cdot w_2^3 + \frac{1}{2} \cdot \left((w_1 + w_2)^3 - w_1^3\right) = 17;$$

thus, the cost shares are not proportional.

Define a function $\Phi$ by

$$\Phi(P) = \sum_{e \in E} \Phi_e(P),$$

where the edge potential $\Phi_e$ is defined as

$$\Phi_e(P) = \sum_{i \in S_e} \chi_{i,e}(S_e^i(\pi) \cup \{i\}) \tag{6}$$

for some ordering $\pi$ on $S_e$. For this definition to make sense, it must be the case that the right-hand side of (6) is independent of the ordering $\pi$. This is a special case of a result of Hart and Mas-Colell [13] (see Section 3.3), for which we give a direct proof.

Proposition 3.2. *For every joint cost function $C$ with player set $S$, the value of*

$$\sum_{i \in S} \mathbf{E}_{\tau^i} \left[ C(S^i(\pi,\tau^i) \cup \{i\}) - C(S^i(\pi,\tau^i)) \right] \tag{7}$$

*is the same for every ordering $\pi$ of $S$, where $\tau^i$ is a permutation of $S^i(\pi) \cup \{i\}$ chosen uniformly at random and $S^i(\pi,\tau^i)$ denotes the players of $S$ that precede $i$ in both $\pi$ and $\tau^i$.*

*Proof.* For a fixed ordering $\pi$ of the players, the quantity in (7) can be written as a sum of the form $\sum_{T \subseteq S} a_T c(T)$ for some set $\{a_T\}_{T \subseteq S}$ of coefficients. We now explicitly compute these coefficients and show that they do not depend on $\pi$.

Fix a subset $T \subseteq S$. With respect to the ordering $\pi$, let $i$ denote the last player of $T$, say in position $\ell$. There is a positive contribution to the coefficient $a_T$ from the $\ell$th summand of (7), and a negative contribution from all subsequent summands. The positive contribution equals the probability that, among all random orderings of the players of $S^i(\pi) \cup \{i\}$, the players of $T$ come first and player $i$ is the last of these. This probability is

$$\frac{(|T|-1)!(\ell - |T|)!}{\ell!}.$$

Let $i_j$ denote the $j$th player in the ordering $\pi$ for some $j > \ell$. The negative contribution to the coefficient $a_T$ by the $j$th summand of (7) equals the probability that, among all random orderings of the first $j$ players under $\pi$, the players of $T$ come first and are immediately followed by player $i_j$. This probability is

$$\frac{|T|!(j-|T|-1)!}{j!}.$$

Summing over all players $j > \ell$ after $T$ under $\pi$ and rewriting, we obtain

$$a_T = (|T|-1)! \left[ \frac{1}{\ell(\ell-1)\cdots(\ell-|T|+1)} - \sum_{j=\ell+1}^{k} \frac{|T|}{j(j-1)\cdots(j-|T|+1)(j-|T|)} \right], \tag{8}$$

where $k$ is the number of players in $S$. Since

$$\frac{|T|}{j(j-1)\cdots(j-|T|+1)(j-|T|)} = \left( \frac{1}{(j-1)\cdots(j-|T|+1)(j-|T|)} - \frac{1}{j(j-1)\cdots(j-|T|+1)} \right)$$

for every $j > \ell$, the sum in (8) telescopes and hence

$$a_T = \frac{(|T|-1)!}{k(k-1)\cdots(k-|T|+1)},$$

which is a function only of the sizes $k$ and $|T|$ and is independent of the position $\ell$ of the final element of $T$ in $\pi$. We conclude that the sum (7) is the same for every ordering $\pi$ of the players $S$.
□

The fact that the function $\Phi$ is a potential function now follows easily.

Proposition 3.3. *For every pair $P$ and$P' = (P_{-i}, P'_i)$ of path vectors that differ only in the $i$th component,*

$$\Phi(P') - \Phi(P) = C_i(P') - C_i(P). \tag{9}$$

*Proof.* As in the proof of Proposition 3.1, we can focus on a single edge $e \in P'_i \setminus P_i$. By Proposition 3.2, we can compute the contribution of $e$ to the left-hand side of (9) using an ordering $\pi$ of the players in which $i$ follows all of the players of $S_e$. Then, edge $e$ contributes exactly $\chi_{i,e}(S_e \cup \{i\})$ to both sides of (9). □

Corollary 3.2. *In every SV atomic selfish routing game, best-response dynamics converges to a PNE.*

**3.3. Arbitrary Congestion-Type Games** The (weighted) Shapley shares in Definitions 3.1 and 3.2 can be generalized to arbitrary congestion-type games. Consider a resource set $E$ and a player set $S = \{1, 2, \ldots, k\}$, where each resource $e$ has a joint cost functions $C_e : 2^S \to \mathbb{R}$ defined on the subsets of $S$, and each player $i$ has a strategy set $\mathcal{P}_i \subseteq 2^E$ and a positive weight $w_i$. For a resource $e$, subset $S_e$ of players, and a player $i \in S_e$, define the weighted Shapley share $\chi_{i,e}(S_e)$ of $i$ for resource $e$ when its users are $S_e$ as its weighted Shapley value (Section 2) in the game with player set $S_e$ and cost function $C_e$ restricted to $2^{S_e}$. The cost $C_i(P)$ to a player $i$ in a strategy profile $P$ is then defined as the sum of its cost shares:

$$C_i(P) = \sum_{e \in P_i} \chi_{i,e}(S_e),$$

where $S_e = \{j \in S : e \in P_j\}$ denotes the users of resource $e$ in the profile $P$.

We claim that every game defined in this way admits a weighted potential function and hence best-response dynamics converges to a PNE. The argument follows that in Section 3.2. Define a function $\Phi = \sum_{e \in E} \Phi_e(P)$ in which the edge potential $\Phi_e$ is defined as

$$\Phi_e(P) = \sum_{i \in S_e} w_i \cdot \chi_{i,e}(S_e^i(\pi) \cup \{i\}) \tag{10}$$

for some ordering $\pi$ on the players $S_e$ using $e$ in $P$. Hart and Mas-Colell [13] proved that the right-hand side of (10) is independent of the ordering $\pi$, for every joint cost functions $C_e$ and positive weight vector $w$. The proof that $\Phi$ is a weighted potential function is the same as in the proof of Proposition 3.3.

**4. The Price of Stability in SV Network Cost-Sharing Games.** This section provides tight bounds on the price of stability in SV network cost-sharing games — the ratio between the cost of the best PNE and the minimum-cost outcome. It is easy to see that, for every weight vector with $k$ players, the worst PNE of such a game can cost $k$ times as much as an optimal solution, and that this is tight [6].

Section 4.1 generalizes a construction of Anshelevich et al. [2] to players with general weights. Section 4.2 is the primary contribution of this section, a matching upper bound for every positive weight vector.

**4.1. POS Lower Bound.** Consider a graph $G = (V, E)$ and players $i = 1, 2, \ldots, k$ with distinct source vertices $s_1, \ldots, s_k$ and a common sink vertex $t$; see also Figure 2. As usual, we assume that $w_1 \le w_2 \le \cdots \le w_k$. The graph has one additional vertex $v$. There is an edge from $v$ to $t$ with cost $1 + \epsilon$, where $\epsilon > 0$ is arbitrarily small. For each $i$, there is a zero-cost edge from $s_i$ to $v$. For each $i$, the edge from $s_i$ to $t$ is set to the weighted Shapley share of the $i$th player for a unit-cost edge shared by players with weights $w_1, w_2, \ldots, w_i$; we denote the quantity by $c_i(\{w_1, w_2, \ldots, w_i\})$.

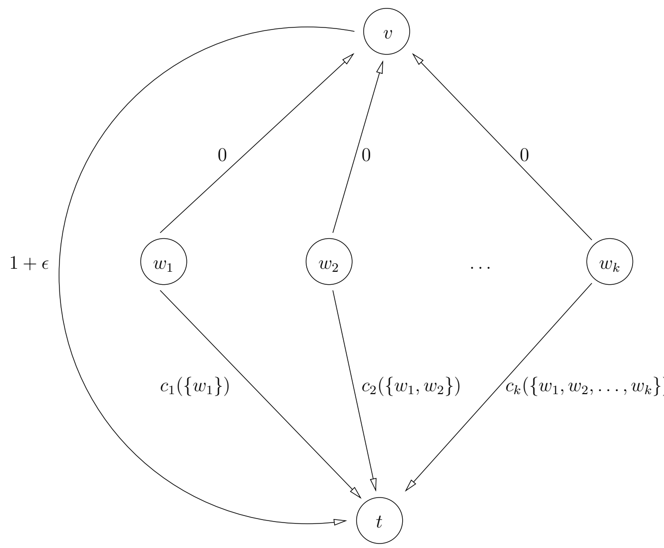



FIGURE 2. Proof of Proposition 4.1. The worst-case price of stability is at least the expression in (11).

In the graph $G$, each player $i$ can either use the path $s_i \to v \to t$, or use the direct edge from $s_i$ to $t$. The optimal solution, in which every player $i$ chooses the path $s_i \to v \to t$, has cost $1+\epsilon$. We claim that in the unique PNE of this SV network cost-sharing game, every player $i$ chooses the direct $s_i$-$t$ edge. To see this, consider the player $k$ with the largest weight. The smallest cost it could have by taking the two-hop path is $(1+\epsilon)\cdot c_k(\{w_1, w_2, \ldots, w_k\})$, which occurs when all players share the edge from $v$ to $t$. This is larger than the cost of its one-hop path. Hence, in every PNE, player $k$ uses its one-hop path and does not share the edge from $v$ to $t$. The same reasoning applies inductively, showing that in every PNE, every player uses its one-hop path. This construction gives the following lower bound for every positive weight vector $w$.

PROPOSITION 4.1. *For every set of $k$ players with positive nondecreasing weight vector $w$, the worst-case price of stability in SV network cost-sharing games with weight vector $w$ is at least*

$$\sum_{i=1}^{k} c_i(\{w_1, w_2, \ldots, w_i\}). \tag{11}$$

Setting $w = (1, 1, \ldots, 1)$ recovers the well-known lower bound of $\mathcal{H}_k$ on the price of stability with unweighted players [2].

**4.2. POS Upper Bound.** The goal of this section is to prove that the lower bound in Proposition 4.1 is tight for every weight vector $w$.

THEOREM 4.1. *For every SV network cost-sharing game with player set $S = \{1, 2, \ldots, k\}$ and positive nondecreasing weight vector $w$, the price of stability is at most*

$$\sum_{i=1}^{k} c_i(\{w_1, w_2, \ldots, w_i\}). \tag{12}$$

The special case of unweighted players, where the bound (12) is the $k$th Harmonic number $\mathcal{H}_k$, has a short proof: the potential function $\Phi$ in (2) is always at least and never more than $\mathcal{H}_k$ times the cost of an outcome, so the potential function minimizer (a PNE) has cost at most $\mathcal{H}_k$ times that of an optimal outcome. With weighted players, the weighted potential function (2) need not approximate the cost of an outcome to any non-trivial factor, and a different argument is called for.

The high-level plan is as follows. We consider a minimum-cost outcome $P^*$ and the outcome $P$ that minimizes the weighted potential function $\Phi$ (2). To bound the cost of $P$ in terms of $P^*$, we transform $P^*$ into $P$ one component at a time, in decreasing order of player weight. The change in outcome cost is the change in the deviating player's cost, which we can bound using the weighted potential function, plus the change in other players' cost. We argue that the worst case occurs when the deviating player abandons all edges it was using previously and switches only to edges that were previously unused. Bounding the cost of this worst case yields the theorem.

Before proceeding to the formal proof of Theorem 4.1, we prove a technical lemma. It states that the upper bound in (12) is nondecreasing in the player set. This is not obvious, as deleting a player removes one summand from (12) but also increases the value of some of the remaining summands.

LEMMA 4.1. *For every set $S=\{1,2,\dots,k\}$ of players with nondecreasing positive weight vector $w$, and every player $j\in S$,*

$$\sum_{i=1}^{k} c_i(\{w_1,w_2,\dots,w_i\}) \geq \sum_{i=1}^{j-1} c_i(\{w_1,w_2,\dots,w_i\}) + \sum_{i=j+1}^{k} c_i(\{w_1,w_2,\dots,w_{j-1},w_{j+1},\dots,w_i\}). \quad (13)$$

*Proof of Lemma 4.1.* The first $j-1$ summands on both sides are the same. Only the left-hand side of (13) has a summand with $i=j$, namely $c_j(\{w_1,w_2,\dots,w_j\})$. For $i>j$, the $i$th summand on the left-hand side $(c_i(\{w_1,w_2,\dots,w_i\}))$ is smaller than the corresponding summand on the right-hand side $(c_i(\{w_1,w_2,\dots,w_{j-1},w_{j+1},\dots,w_i\}))$. To calculate the difference, we turn to the probabilistic representation of weighted Shapley shares in terms of exponentially distributed random variables $\mathbf{X}_1,\dots,\mathbf{X}_k$ (Section 3.1).

In the right-hand side summand, the random variable $\mathbf{X}_i$ does not have to compete with $\mathbf{X}_j$ in order to be the largest. Hence, the event that $\mathbf{X}_i$ is smaller than $\mathbf{X}_j$ but larger than every other player's random variable contributes to $c_i(\{w_1,w_2,\dots,w_{j-1},w_{j+1},\dots,w_i\})$ but not to $c_i(\{w_1,w_2,\dots,w_i\})$. We denote this probability by $p_j(i)$. Recalling the density and distribution functions of exponentially distributed random variables, we have

$$p_j(i)=\int_0^\infty \lambda_i e^{-\lambda_i x} e^{-\lambda_j x} \prod_{l=1}^{j-1}\left(1-e^{-\lambda_l x}\right) \prod_{l=j+1}^{i-1}\left(1-e^{-\lambda_l x}\right) dx.$$

Recalling that $w_1\leq w_2\leq\cdots\leq w_k$ and hence $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k$, the difference $\Delta$ between the left-hand and right-hand sides of (13) is

$$\begin{aligned}
\Delta &= c_j(\{w_1,w_2,\dots,w_j\}) - \sum_{i=j+1}^{k} p_j(i)\\
&= \int_0^\infty \lambda_j e^{-\lambda_j x}\prod_{l=1}^{j-1}(1-e^{-\lambda_l x}) - \sum_{i=j+1}^{k}\lambda_i e^{-\lambda_i x} e^{-\lambda_j x}\prod_{l=1,l\neq j}^{i-1}(1-e^{-\lambda_l x})dx\\
&\geq \int_0^\infty \lambda_j e^{-\lambda_j x}\prod_{l=1}^{j-1}(1-e^{-\lambda_l x})\left[1-\sum_{i=j+1}^{k} e^{-\lambda_j x}\prod_{h=j+1}^{i-1}(1-e^{-\lambda_h x})\right]\\
&\geq \int_0^\infty \lambda_j e^{-\lambda_j x}\prod_{l=1}^{j-1}(1-e^{-\lambda_l x})\left[1-\underbrace{e^{-\lambda_j x}\sum_{i=j+1}^{k}(1-e^{-\lambda_j x})^{i-j-1}}_{\leq e^{-\lambda_j x}\cdot e^{\lambda_j x}=1}\right]\\
&\geq 0.
\end{aligned}$$

This concludes the proof of the lemma. □

We now prove Theorem 4.1

*Proof of Theorem 4.1.* Let $P^*$ and $P$ denote a minimum-cost outcome and an outcome that minimizes the weighted potential function $\Phi$ in (2), respectively. For the analysis, we imagine each player $i$ deviating from $P_i^*$ to $P_i$ in nonincreasing weight order, i.e., in the order $k, k-1, \ldots, 1$. Let $T_e^i$ denote the players using edge $e$ before player $i$ switches strategies, and let $\Delta\Phi_i$ denote the change in $\Phi$ when $i$ switches strategies. By Proposition 3.1, the change in player $i$'s cost is exactly $(\Delta\Phi_i)/w_i$. To compute the change in other players' costs, recall that the sum of the weighted Shapley shares of an edge used by at least one player always equals the cost of that edge. Thus, for every edge $e \in P_i^* \setminus P_i$ with $|T_e^i| \geq 2$, player $i$'s withdrawal from edge $e$ increases the sum of the cost shares of the players of $T_e^i \setminus \{i\}$ by $\chi_{i,e}(T_e^i)$. Symmetrically, for every edge $e \in P_i \setminus P_i^*$, player $i$'s arrival to edge $e$ decreases the sum of cost shares of players in $T_e^i$ (if any) by $\chi_{i,e}(T_e^i \cup \{i\})$. Overall, when player $i$ switches from $P_i^*$ to $P_i$, the outcome cost increases by at most

$$\frac{\Delta\Phi_i}{w_i} + \sum_{e \in P_i^* \setminus P_i \,:\, |T_e^i| \geq 2} \chi_{i,e}(T_e^i).$$

Summing over all players $i$, we obtain

$$C(P) - C(P^*) \leq \sum_{i=1}^{k} \frac{\Delta\Phi_i}{w_i} + \sum_{i=1}^{k} \sum_{e \in P_i^* \setminus P_i \,:\, |T_e^i| \geq 2} \chi_{i,e}(T_e^i). \tag{14}$$

To bound the first term of the right-hand side of (14), write

$$\sum_{i=1}^{k} \frac{\Delta\Phi_i}{w_i} = \underbrace{\frac{1}{w_k} \sum_{j=1}^{k} \Delta\Phi_j}_{\leq 0} + \sum_{i=1}^{k-1} \left( \underbrace{\frac{1}{w_i} - \frac{1}{w_{i+1}}}_{\geq 0} \right) \underbrace{\sum_{j=1}^{i} \Delta\Phi_j}_{\leq 0}.$$

Since $w_1 \leq w_2 \leq \cdots \leq w_k$, every term $(\frac{1}{w_i} - \frac{1}{w_{i+1}})$ is nonnegative. Every term $\sum_{j=1}^{i} \Delta\Phi_j$ is the total potential function change of a sequence of moves that terminates in the outcome $P$ that minimizes $\Phi$, and hence is nonpositive. We conclude that the term $\sum_i \frac{\Delta\Phi_i}{w_i}$ in (14) is nonpositive.

We next upper bound the contribution of each edge $e$ to the second term in (14). Let $S_e^*$ denote the set of players that use $e$ in $P^*$, and $S_e^i = S_e^* \cap \{1, 2, \ldots, i\}$. We claim that

$$\sum_{i \,:\, e \in P_i^* \setminus P_i, |T_e^i| \geq 2} \chi_{i,e}(T_e^i) \leq \sum_{i \in S_e^* \,:\, |S_e^i| \geq 2} \chi_{i,e}(S_e^i). \tag{15}$$

The right-hand side of (15) corresponds to the scenario in which every user of $e$ in $P^*$ abandons $e$ when switching to its strategy in $P$.

Inequality (15) follows from three observations. First, for each $\ell = 2, 3, \ldots, |S_e^*|$, the right-hand side of (15) contains exactly one summand $\chi_{i,e}(S_e^i)$ in which $|S_e^i| = \ell$. The corresponding set $S_e^i$ contains the $\ell$ lowest-indexed — and hence lowest-weight — players of $S_e^*$, of which $i$ has maximum weight. Second, for each $\ell = 2, 3, \ldots, |S_e^*|$, the left-hand side of (15) contains at most one summand $\chi_{j,e}(T_e^j)$ in which $|T_e^j| = \ell$. The corresponding set $T_e^j$ contains $h \geq 0$ players with index higher than $j$, who have already deviated to another path that contains $e$, and the $\ell - h$ lowest-weight players of $S_e^*$, of which $j$ has maximum weight. Third, Definition 3.1 implies that the weighted Shapley share of a player is increasing in its own weight and decreasing in other players' weights. Thus, for every $\ell = 2, 3, \ldots, |S_e^*|$, the summand on the right-hand side of (15) with $|S_e^i| = \ell$ is at least the summand on the left-hand side with $|T_e^j| = \ell$ (if any).

Combining our inequalities, applying Lemma 4.1, and using the fact that $C(P^*) = \sum_{e\,:\,S_e^* \neq \emptyset} \gamma_e$, we have

$$\begin{aligned} C(P) - C(P^*) &\leq \sum_{e \in E} \sum_{i \in S_e^*\,:\,|S_e^i| \geq 2} \chi_{i,e}(S_e^i) \\ &= \sum_{e \in E\,:\,S_e^* \neq \emptyset} \left( \sum_{i \in S_e^*} \chi_{i,e}(S_e^i) - \gamma_e \right) \\ &\leq \sum_{e \in E\,:\,S_e^* \neq \emptyset} \gamma_e \left( \sum_{i=1}^{k} c_i(w_1, w_2, \ldots, w_i) - 1 \right), \end{aligned}$$

and hence

$$C(P) \leq C(P^*) \cdot \sum_{i=1}^{k} c_i(w_1, w_2, \ldots, w_i),$$

which proves the theorem. □

As a special case, if players' weights are within a constant factor of each other, then $c_i(w_1, w_2, \ldots, w_i) = \Theta(1/i)$ for every $i$ and the POS is $O(\log k)$. In contrast, when PNE exist under proportional cost shares, the POS in this case can be $\Theta(k)$ [5].

More generally, the POS bound in Proposition 4.1 and Theorem 4.1 approaches $k$ as the players' weights become more dramatically spread out. For example, when $w_i = i$ for $i = 1, 2, \ldots, k$, calculations show that the POS is $O(\sqrt{k})$.

**5. The Price of Anarchy in SV Atomic Selfish Routing Games.** This section gives matching upper and lower bounds on the worst-case price of anarchy in SV atomic selfish routing games. Section 5.1 covers preliminaries. Section 5.2 proves a POA upper bound that is parameterized by the set of resource cost functions. Section 5.3 evaluates this upper bound for cost functions that are polynomials with nonnegative coefficients. Section 5.4 gives a construction showing that, for every set of cost functions satisfying some mild technical conditions, this POA upper bound is tight in the worst case.

**5.1. Preliminaries.** The worst-case POA in SV atomic selfish routing games depends on the set of allowable cost functions. For example, with cost functions that are polynomials with degree at most $d$ and nonnegative coefficients, we prove that the worst-case POA is exponential in $d$, but independent of the network size and the number of players. This dependence motivates parameterizing our POA bounds by the class $\mathcal{C}$ of allowable resource cost functions. We do not expect the worst-case POA to admit a closed-form expression for every set $\mathcal{C}$, and instead seek a relatively simple characterization of this value. Throughout this section, we make the following assumptions.

1. Every cost function $c \in \mathcal{C}$ is nonnegative and nondecreasing.
2. For every $c \in \mathcal{C}$ and $w \geq 0$, $c(x+w)(x+w) - c(x)x$ is a convex and nondecreasing function of $x$. This condition holds if, for example, the function $c$ is twice differentiable with nondecreasing first and second derivatives.
3. The set $\mathcal{C}$ is closed under scaling and dilation, meaning that if $c(x) \in \mathcal{C}$ and $a, b > 0$, then $a \cdot c(bx) \in \mathcal{C}$.

**5.2. POA Upper Bound.** Our upper bound approach is an instantiation of the "smoothness framework" articulated in [24]. We call a pair $(\lambda, \mu)$ of real numbers *feasible* for a cost function $c$ if $\mu < 1$ and if

$$\frac{1}{2}c(x+x^*)(x+x^*) + \frac{1}{2}c(x)x + \frac{1}{2}c(x^*)x^* \leq \lambda c(x^*)x^* + \mu c(x)x \tag{16}$$

for every $x, x^* \geq 0$. We use $\mathcal{A}(\mathcal{C})$ to denote the set of pairs $(\lambda, \mu)$ that are feasible for every cost function $c \in \mathcal{C}$. Define

$$\zeta(\mathcal{C}) = \inf_{(\lambda,\mu)\in\mathcal{A}(\mathcal{C})} \tfrac{\lambda}{1-\mu},$$

or as $+\infty$ if $\mathcal{A}(\mathcal{C}) = \emptyset$.

Under the assumptions described in Section 5.1, $\zeta(\mathcal{C})$ is an upper bound on the POA of every SV atomic selfish routing game with cost functions in $\mathcal{C}$.

Theorem 5.1. *Let $\mathcal{C}$ be a set of nonnegative, nondecreasing cost functions with $c(x+w)(x+w) - c(x)x$ convex and nondecreasing in $x$ for every $w \geq 0$ and $c \in \mathcal{C}$. Then the POA of every SV atomic selfish routing game with cost functions in $\mathcal{C}$ is at most $\zeta(\mathcal{C})$.*

*Proof.* Let $P$ and $P^*$ denote a PNE and an arbitrary outcome of such a routing game with players $S = \{1, 2, \ldots, k\}$. Since $P$ is a PNE, we have

$$\begin{aligned} C(P) &= \sum_{i=1}^{k} \sum_{e\in P_i} \chi_{i,e}(S_e) \\ &\leq \sum_{i=1}^{k} \sum_{e\in P_i^*} \chi_{i,e}(S_e \cup \{i\}), \end{aligned} \tag{17}$$

where $S_e$ denotes the players using edge $e$ in $P$. By Definition 3.2,

$$\chi_{i,e}(S_e \cup \{i\}) = \mathbf{E}[(\mathbf{X}_{i,e} + w_i) \cdot c_e(\mathbf{X}_{i,e} + w_i) - \mathbf{X}_{i,e} \cdot c_e(\mathbf{X}_{i,e})], \tag{18}$$

where $w_i$ is the weight of player $i$ and $\mathbf{X}_{i,e}$ is the total weight of the players preceding $i$ in a uniformly random ordering of the players in $S_e \cup \{i\}$.

Let $x_e$ denote the total weight of the players in $S_e$. Pairing up subsets of $S_e$ with their complements, the right-hand side of (18) is a convex combination of terms of the form $\frac{1}{2}[c_e(z+w_i)(z+w_i) - c_e(z)z] + \frac{1}{2}[c_e((x_e-z)+w_i)((x_e-z)+w_i) - c_e(x_e-z)(x_e-z)]$. Since $c_e(x+w_i)(x+w_i) - c_e(x)x$ is assumed convex and nondecreasing in $x$, each of these terms is maximized when $z = x_e$. Thus,

$$\begin{aligned} C(P) &\leq \sum_{i=1}^{k} \sum_{e\in P_i^*} [\tfrac{1}{2}(c_e(x_e+w_i)(x_e+w_i) - c_e(x_e)x_e) + \tfrac{1}{2}c_e(w_i)w_i] \\ &\leq \sum_{e\in E} [\tfrac{1}{2}(c_e(x_e+x_e^*)(x_e+x_e^*) - c_e(x_e)x_e) + \tfrac{1}{2}c_e(x_e^*)x_e^*], \end{aligned}$$

where $x_e^*$ denotes the total weight of players using edge $e$ in $P^*$, with the second inequality following from the fact that the function $c(x+w)(x+w) - c(x)x$ is superadditive in $w$ for every fixed $x$.

Now choose $(\lambda, \mu) \in \mathcal{A}(\mathcal{C})$. Since $(\lambda, \mu)$ satisfies (16) for all $c \in \mathcal{C}$ and $x, x^* \geq 0$, we have

$$\begin{aligned} C(P) &\leq \sum_{e\in E} [\tfrac{1}{2}(c_e(x_e+x_e^*)(x_e+x_e^*) - c_e(x_e)x_e) + \tfrac{1}{2}c_e(x_e^*)x_e^*] \\ &\leq \sum_{e\in E} [\lambda c_e(x_e^*)(x_e^*) + \mu c_e(x_e)x_e] \\ &= \lambda C(P^*) + \mu C(P); \end{aligned}$$

rearranging terms completes the proof. □

Remark 5.1. The POA upper bound in Theorem 5.1 is a "smoothness proof" in the sense of Roughgarden [24]. Informally, this means that the hypothesis that $P$ is a PNE is used only in the inequality (17), with hypothetical deviations $P_i^*$ that are independent of the choice of $P$. This fact is interesting because POA bounds that are proved with smoothness arguments extend automatically to numerous other equilibrium concepts. Specifically, the POA upper bound of $\zeta(\mathcal{C})$ applies more generally to mixed-strategy Nash equilibria, correlated equilibria, and outcome sequences generated by no-regret learners [24]. Approximate Nash equilibria and polynomial-length best-response sequences also approximately obey the $\zeta(\mathcal{C})$ bound [24]. Finally, the POA bound of $\zeta(\mathcal{C})$ extends to all Bayes-Nash equilibria of incomplete information SV selfish routing games, where players' weights and source-sink pairs are drawn from an arbitrary product prior distribution [25, 29].

**5.3. Example: Polynomial Cost Functions.** This section explicitly evaluates the POA upper bound in Theorem 5.1 for the special case in which $\mathcal{C}$ is the set of polynomials with nonnegative coefficients and maximum degree $d$.

Elementary calculus shows that, for every positive integer $d$, the function

$$g_d(x) = 3x^{d+1} - 1 - (x+1)^{d+1} \tag{19}$$

has a unique positive root, which we denote by $\chi_d$. This section establishes the following theorem.

THEOREM 5.2. *If $\mathcal{C}$ is the set of polynomials with nonnegative coefficients and maximum degree $d$, then the POA of a SV atomic selfish routing game with cost functions in $\mathcal{C}$ is at most $\chi_d^{d+1} = (\Theta(d))^{d+1}$.*

Remark 5.2 shows that the bound of $\chi_d^{d+1}$ is tight in the worst case, for every positive integer $d$. For comparison, the worst-case POA with proportional (rather than SV) cost-sharing, in such games that happen to possess PNE, is the slightly smaller quantity $\Theta((d/\ln d)^{d+1})$.

Before presenting the proof of Theorem 5.2, we examine the asymptotic behavior of $\chi_d$.

PROPOSITION 5.1. *As $d \to \infty$, $\chi_d = \Theta(d)$.*

*Proof.* Note that

$$g_d(d) = 3d^{d+1} - 1 - (d+1)^{d+1} = 3d^{d+1} - 1 - d^{d+1}(1+1/d)(1+1/d)^d.$$

Similarly,

$$g(d/2) = 3(d/2)^{d+1} - 1 - (d/2)^{d+1}(1+2/d)\left((1+2/d)^{d/2}\right)^2.$$

Since $\lim_{x\to\infty}(1+1/x)^x = e$,

$$\lim_{d\to\infty} g_d(d) > 0 \text{ and } \lim_{d\to\infty} g_d(d/2) < 0.$$

Since $g_d$ is increasing on $[1,\infty)$, $\chi_d \in (d/2, d)$ for all sufficiently large $d$. More careful computations of this type show that $\chi_d$ tends to infinity as roughly $0.9d$. $\square$

We now prove Theorem 5.2.

*Proof of Theorem 5.2.* We exhibit values $(\lambda, \mu)$ that are feasible for every cost function $c \in \mathcal{C}$ — recall (16) — and that satisfy $\lambda/(1-\mu) \le \chi_d^{d+1}$. The theorem then follows from Theorem 5.1.

Define

$$\lambda_j = \frac{(\chi_j+1)^j + 1}{2} \quad \text{and} \quad \mu_j = \frac{\left(\chi_j^{-1}+1\right)^j - 1}{2} \quad \text{for} \quad j = 1, 2, \ldots, d,$$

$\lambda = \max_j \lambda_j$, and $\mu = \max_j \mu_j$. It is evident that $\lambda = \lambda_d$.

We begin by showing that $(\lambda, \mu)$ is feasible. To see that $\mu < 1$, recall that, by definition,

$$3\chi_j^{j+1} = 1 + (\chi_j+1)^{j+1}. \tag{20}$$

If $\mu_j \geq 1$, then $(\chi_j^{-1}+1)^j \geq 3$, which implies that $(1+\chi_j)^j \geq 3\chi_j^j$ and hence $(1+\chi_j)^{j+1} \geq 3\chi_j^j + 3\chi_j^{j+1}$. Combining this with (20) yields the contradiction $3\chi_j^j \leq -1$.

To see that $(\lambda, \mu)$ satisfies (16) for every cost function $c \in \mathcal{C}$, fix such a function $c(x) = \sum_{j=0}^d a_j x^j$. By linearity, the condition (16) reduces to proving that

$$\frac{(x+x^*)^{j+1}}{2} - \frac{x^{j+1}}{2} + \frac{(x^*)^{j+1}}{2} \leq \lambda (x^*)^{j+1} + \mu x^{j+1}. \tag{21}$$

for every $j = 1, 2, \ldots, d$ and $x, x^* \geq 0$. Every $\lambda_n, \mu_n$ pair clearly satisfies inequality (21) when $x^* = 0$. Assume that $x^* > 0$ and set $r = x/x^*$. Rewrite inequality (21) as

$$(2\mu_j + 1) r^{j+1} - (1+r)^{j+1} + (2\lambda_j - 1) \geq 0, \text{ for all } r \geq 0. \tag{22}$$

Considering the left-hand side of (22) as a function of $r$ and taking the derivative, we can see that the minimizer is $r = \left((2\mu+1)^{1/j} - 1\right)^{-1} = \chi_j$. With these values of $r, \lambda_j, \mu_j$, the left-hand side of (22) equals 0, which verifies inequality (22) (and (21)). Inequality (21) clearly remains valid for $\lambda \geq \lambda_j$ and $\mu \geq \mu_j$, and so $(\lambda, \mu)$ form a feasible pair.

To prove that $\lambda/(1-\mu) \leq \chi_d^{d+1}$, recall that $\lambda = \lambda_d$ and write $\mu = \mu_\ell$ for some $\ell \in \{1, 2, \ldots, d\}$. Then,

$$\begin{aligned}
\frac{\lambda}{1-\mu} &= \frac{(\chi_d + 1)^d + 1}{3 - (\chi_\ell^{-1} + 1)^\ell} \\
&= \frac{1}{3} \frac{3\chi_\ell^{\ell+1}\left((\chi_d+1)^d + 1\right)}{3\chi_\ell^{\ell+1} - \chi_\ell(\chi_\ell+1)^\ell} \\
&= \frac{1}{3} \frac{(\chi_\ell+1)^{\ell+1} + 1}{(\chi_\ell+1)^\ell + 1} \left((\chi_d+1)^d + 1\right),
\end{aligned}$$

where the last step follows from (20). The last expression is clearly increasing in $\ell$. Hence, setting $\ell = d$ and using (20) once again, we derive $\lambda/(1-\mu) \leq \chi_d^{d+1}$, as required. □

**5.4. POA Lower Bound.** The upper bounds presented in Section 5.2 are tight in the worst case. The construction that proves this is simplest to present in the context of general congestion games where players have strategy sets that are arbitrary subsets of the edges and not necessarily paths. It is not difficult to convert the construction into an atomic selfish routing network.

THEOREM 5.3. *For every class $\mathcal{C}$ that is closed under scaling and dilation, the POA of a SV atomic congestion game with cost functions in $\mathcal{C}$ can be arbitrarily close to $\zeta(\mathcal{C})$.*

Our construction resembles one used previously to prove POA lower bounds for weighted congestion games with proportional cost shares [4], but some of the technical details differ. Our proof of Theorem 5.3 requires the following technical lemma. It identifies the cost functions and the equilibrium and optimal edge loads that are the necessary ingredients in any worst-case example.

LEMMA 5.1. *Let $\mathcal{C}$ be a class of cost functions with $\zeta(\mathcal{C}) > 1$. For every positive $\epsilon < \zeta(\mathcal{C}) - 1$, at least one of the following conditions holds.*

1. *There exist $c \in \mathcal{C}, x \geq 0, x^* > 0$ such that*

$$\frac{1}{2} \cdot (x+x^*) \cdot c(x+x^*) - \frac{1}{2} \cdot x \cdot c(x) + \frac{1}{2} \cdot x^* \cdot c(x^*) \geq x \cdot c(x)$$

*and*

$$\frac{x \cdot c(x)}{x^* \cdot c(x^*)} \geq \zeta(\mathcal{C}) - \epsilon.$$

2. *There exist* $c_1, c_2 \in \mathcal{C}$, $x_1, x_2 \geq 0$, $x_1^*, x_2^* > 0$, *and* $\lambda, \mu$ *such that*

$$\frac{1}{2}\cdot(x_1+x_1^*)\cdot c_1(x_1+x_1^*) - \frac{1}{2}\cdot x_1\cdot c_1(x_1) + \frac{1}{2}\cdot x_1^*\cdot c_1(x_1^*) = \lambda\cdot x_1^*\cdot c_1(x_1^*) + \mu\cdot x_1\cdot c_1(x_1);$$

$$\frac{1}{2}\cdot(x_2+x_2^*)\cdot c_2(x_2+x_2^*) - \frac{1}{2}\cdot x_2\cdot c_2(x_2) + \frac{1}{2}\cdot x_2^*\cdot c_2(x_2^*) = \lambda\cdot x_2^*\cdot c_2(x_2^*) + \mu\cdot x_2\cdot c_2(x_2);$$

$$\frac{1}{2}\cdot(x_1+x_1^*)\cdot c_1(x_1+x_1^*) - \frac{1}{2}\cdot x_1\cdot c_1(x_1) + \frac{1}{2}\cdot x_1^*\cdot c_1(x_1^*) \leq x_1\cdot c_1(x_1);$$

$$\frac{1}{2}\cdot(x_2+x_2^*)\cdot c_2(x_2+x_2^*) - \frac{1}{2}\cdot x_2\cdot c_2(x_2) + \frac{1}{2}\cdot x_2^*\cdot c_2(x_2^*) \geq x_2\cdot c_2(x_2);$$

*and*

$$\frac{\lambda}{1-\mu} > \zeta(\mathcal{C}) - \epsilon.$$

*Proof.* For a cost function $c \in \mathcal{C}$, $x \geq 0$, and $x^* > 0$, let $\mathcal{H}_{c,x,x^*}$ denote the half-plane

$$\frac{1}{2}\cdot(x+x^*)\cdot c(x+x^*) - \frac{1}{2}\cdot x\cdot c(x) + \frac{1}{2}\cdot x^*\cdot c(x^*) \leq \lambda\cdot x^*\cdot c(x^*) + \mu\cdot x\cdot c(x)$$

and $\partial\mathcal{H}_{c,x,x^*}$ the boundary of this half-plane. Recall from (16) that these are the half-planes that define the set $\mathcal{A}(\mathcal{C})$ of feasible pairs $(\lambda,\mu)$ for the set $\mathcal{C}$ of cost functions. Also, define

$$\beta_{c,x,x^*} = \frac{x\cdot c(x)}{\frac{1}{2}\cdot(x+x^*)\cdot c(x+x^*) - \frac{1}{2}\cdot x\cdot c(x) + \frac{1}{2}\cdot x^*\cdot c(x^*)}$$

and

$$\zeta_{c,x,x^*} = \frac{x\cdot c(x)}{x^*\cdot c(x^*)}.$$

Fix a positive $\epsilon < \zeta(\mathcal{C}) - 1$ and let $\zeta' = \zeta(\mathcal{C}) - \epsilon/2$. If $\zeta(\mathcal{C})$ is not finite, set $\zeta' = 1/\epsilon$. We write $\mathcal{L}_{\zeta'}$ for the line $\lambda + \zeta'\cdot\mu = \zeta'$ in the $\lambda,\mu$ plane.

If we think of a boundary line $\partial\mathcal{H}_{c,x,x^*}$ as specifying $\mu$ as a function of $\lambda$, then this line has slope $-1/\zeta_{c,x,x^*}$ and $\mu$-intercept $1/\beta_{c,x,x^*}$. The half-space $\mathcal{H}_{c,x,x^*}$ consists of everything "northeast" of its boundary.

Consider the half-planes with $\beta_{c,x,x^*} \leq 1$. In the lucky event that there is such a half-plane with $\zeta_{c,x,x^*} \geq \zeta'$, we are done: this choice of $c, x, x^*$ satisfies the conditions of the first case of the lemma. For the rest of the proof, we assume that $\zeta_{c,x,x^*} < \zeta'$ for every half-plane with $\beta_{c,x,x^*} \leq 1$.

We consider two cases. To define them, pick an arbitrary cost function $c_1$ with $c_1(1) > 0$ — since $\mathcal{C}$ is closed under dilation, such a function exists — and a sufficiently large value of $x_1$ so that $\zeta_{c_1,x_1,1} > \zeta'$. Our standing assumption implies that $\beta_{c_1,x_1,1} > 1$. Define $(\hat{\lambda},\hat{\mu})$ as the unique point of intersection of $\partial H_{c_1,x_1,1}$ and $\mathcal{L}_{\zeta'}$. Since the former line has a larger slope ($-1/\zeta_{c_1,x_1,1}$ vs.$-1/\zeta'$) and a smaller $\mu$-intercept ($1/\beta_{c_1,x_1,1}$ vs. 1) than the latter, $\hat{\lambda} > 0$ and hence $\hat{\mu} < 1$.

For the first case, we assume that there exists a half-plane $\mathcal{H}_{c_2,x_2,x_2^*}$ with $\beta_{c_2,x_2,x_2^*} < 1$ whose boundary intersects the line $\mathcal{L}_{\zeta'}$ at a point $(\lambda_2,\mu_2)$ with $\mu_2 < \hat{\mu}$. Equivalently, the line $\partial\mathcal{H}_{c_2,x_2,x_2^*}$ intersects $\mathcal{L}_{\zeta'}$ to the right of where $\partial\mathcal{H}_{c_1,x_1,1}$ intersects $\mathcal{L}_{\zeta'}$. Since the $\mu$-intercepts of $\partial\mathcal{H}_{c_2,x_2,x_2^*}$ and $\partial\mathcal{H}_{c_1,x_1,1}$ (namely, $1/\beta_{c_2,x_2,x_2^*} > 1$ and $1/\beta_{c_1,x_1,1} < 1$) are on either side of that of $L_{\zeta'}$ (namely, 1) and $\hat{\lambda} > 0$, this implies that the intersection $(\lambda,\mu)$ of $\partial\mathcal{H}_{c_1,x_1,1}$ and $\partial\mathcal{H}_{c_2,x_2,x_2^*}$ lies on the "northeast side" of $L_{\zeta'}$. It follows that $\lambda + \zeta'\mu \geq \zeta'$. Thus, $c_1, c_2, x_1, x_2, 1, x_2^*, \lambda, \mu$ satisfy the conditions in the second case of the lemma.

Finally, assume that all half-planes $\mathcal{H}_{c,x,x^*}$ with $\beta_{c,x,x^*} < 1$ have boundaries that intersect the line $\mathcal{L}_{\zeta'}$ at points $(\lambda, \mu)$ with $\mu \geq \hat{\mu}$. Let $\mu^*$ denote the infimum of all $\mu$-coordinates of such intersections. Under our standing assumption, every such boundary $\partial\mathcal{H}_{c,x,x^*}$ has a smaller slope ($-1/\zeta_{c,x,x^*}$ vs.$-1/\zeta'$) and a larger $\mu$-intercept ($1/\beta_{c_1,x_1,1}$ vs. 1) than $\mathcal{L}_{\zeta'}$, and hence intersects $\mathcal{L}_{\zeta'}$ at a point $(\lambda, \mu)$ with $1 > \mu \geq \hat{\mu}$. Thus, $1 > \mu^* \geq \hat{\mu}$.

We now find appropriate $(c_1, x_1, x_1^*)$ and $(c_2, x_2, x_2^*)$ with $\beta_{c_1,x_1,x_1^*} \geq 1$ and $\beta_{c_2,x_2,x_2^*} < 1$, such that the corresponding half-plane boundaries intersect $\mathcal{L}_{\zeta'}$ at points $(\lambda_1, \mu_1)$ and $(\lambda_2, \mu_2)$ with $\mu_1, \mu_2$ very close to $\mu^*$. Let $\delta = \frac{\epsilon\cdot(1-\mu^*)}{4\cdot\zeta'-\epsilon} > 0$. Consider the point $(\zeta' \cdot (1 - \mu^* + \delta), \mu^* - \delta)$ of $\mathcal{L}_{\zeta'}$. This point is feasible for all constraints corresponding to $(c, x, x^*)$ with $\beta_{c,x,x^*} < 1$. Since $\zeta' < \zeta(\mathcal{C})$, this point cannot belong to the feasible set $\mathcal{A}(\mathcal{C})$ and hence there exists $(c_1, x_1, x_1^*)$ with $\beta_{c_1,x_1,x_1^*} \geq 1$ such that the point $(\zeta' \cdot (1 - \mu^* + \delta), \mu^* - \delta)$ violates the corresponding constraint. Note that the point $(0, 1)$ of $\mathcal{L}_{\zeta'}$ lies in $\mathcal{H}_{c_1,x_1,x_1^*}$. This implies that $\partial\mathcal{H}_{c_1,x_1,x_1^*}$ intersects $\mathcal{L}_{\zeta'}$ at a point $(\lambda_1, \mu_1)$ with $\mu_1 \geq \mu^* - \delta$. Moreover, $\lambda_1 + \zeta' \cdot \mu_1 = \zeta'$.

If $\mu_1 > \mu^*$, then we can find $(c_2, x_2, x_2^*)$ with $\beta_{c_2,x_2,x_2^*} < 1$ that intersects $\mathcal{L}_{\zeta'}$ at $(\lambda_2, \mu_2)$ with $\mu^* \leq \mu_2 \leq \mu_1$. Then, similarly to the previous case, $\partial\mathcal{H}_{c_1,x_1,x_1^*}$ and $\partial\mathcal{H}_{c_2,x_2,x_2^*}$ intersect at a point $(\lambda, \mu)$ such that $\lambda/(1-\mu) \geq \zeta'$, completing the proof.

We can now assume that $\mu^* - \delta \leq \mu_1 \leq \mu^*$. By the definition of $\mu^*$, there exist $(c_2, x_2, x_2^*)$ such that $\partial\mathcal{H}_{c_2,x_2,x_2^*}$ intersects $\mathcal{L}_{\zeta'}$ at $(\lambda_2, \mu_2)$, with $\mu^* \leq \mu_2 \leq \mu^* + \delta$. Note that $\mu_2 \geq \mu_1$ and $\lambda_2 + \zeta' \cdot \mu_2 = \zeta'$.

Let $(\lambda, \mu)$ be the point where $\partial\mathcal{H}_{c_1,x_1,x_1^*}$ and $\partial\mathcal{H}_{c_2,x_2,x_2^*}$ intersect. Both these boundaries have negative slopes, which means $(\lambda, \mu)$ lies in the triangle formed by the points $(\lambda_1, \mu_1)$, $(\lambda_2, \mu_2)$, and $(\lambda_2, \mu_1)$. Then $\lambda/(1-\mu) \geq \lambda_2/(1-\mu_1)$. Since $\lambda_1 - \lambda_2 = \zeta' \cdot (\mu_2 - \mu_1) \leq 2 \cdot \zeta' \cdot \delta$, we have

$$\begin{aligned}\frac{\lambda_2}{1-\mu_1} &= \frac{\lambda_1}{1-\mu_1} - \frac{\lambda_1 - \lambda_2}{1-\mu_1} \\ &\geq \zeta' - \frac{2\cdot\zeta'\cdot\delta}{1-\mu^*+\delta} \\ &\geq \zeta' - \frac{\epsilon}{2}.\end{aligned}$$

This proves that the conditions of the second case in the statement of the lemma hold. □

Before proceeding with the proof of Theorem 5.3, we take note of some consequences of Lemma 5.1. Suppose the second case of the lemma applies and offers two triples $(c_1, x_1, x_1^*)$ and $(c_2, x_2, x_2^*)$ such that the corresponding half-plane boundaries intersect at $(\lambda, \mu)$ with $\lambda/(1-\mu) > \zeta(\mathcal{C}) - \epsilon$. Scaling and dilating a cost function does not affect the corresponding constraint (16). Thus, for every $w > 0$, we can find cost functions $\hat{c}_1$ and $\hat{c}_2$ such that

$$\begin{aligned}\frac{1}{2}\cdot\hat{c}_1(w\cdot(z_1+1))\cdot(z_1+1) &= \left(\lambda - \frac{1}{2}\right)\cdot\hat{c}_1(w) + \left(\mu + \frac{1}{2}\right)\cdot z_1\cdot\hat{c}_1(w\cdot z_1); \\ \frac{1}{2}\cdot\hat{c}_2(w\cdot(z_2+1))\cdot(z_2+1) &= \left(\lambda - \frac{1}{2}\right)\cdot\hat{c}_2(w) + \left(\mu + \frac{1}{2}\right)\cdot z_2\cdot\hat{c}_2(w\cdot z_2),\end{aligned} \tag{23}$$

where $z_1 = x_1/x_1^*$ and $z_2 = x_2/x_2^*$.

Moreover, since $\frac{1}{2}\cdot(1+z_1)\cdot\hat{c}_1(w\cdot(1+z_1)) - \frac{1}{2}\cdot z_1\cdot\hat{c}_1(w\cdot z_1) + \frac{1}{2}\cdot\hat{c}_1(w) \leq z_1\cdot\hat{c}_1(z_1\cdot w)$ and $\frac{1}{2}\cdot(1+z_2)\cdot\hat{c}_2(w\cdot(1+z_2)) - \frac{1}{2}\cdot z_2\cdot\hat{c}_2(w\cdot z_2) + \frac{1}{2}\cdot\hat{c}_2(w) \geq z_2\cdot\hat{c}_2(z_2\cdot w)$, there is a constant $\eta \in [0,1]$ such that

$$\begin{aligned}&\eta\cdot z_1\cdot\hat{c}_1(w\cdot z_1) + (1-\eta)\cdot z_2\cdot\hat{c}_2(w\cdot z_2) = \\ &\eta\cdot\left[\frac{1}{2}\cdot(1+z_1)\cdot\hat{c}_1(w\cdot(1+z_1)) - \frac{1}{2}\cdot z_1\cdot\hat{c}_1(w\cdot z_1) + \frac{1}{2}\cdot\hat{c}_1(w)\right] + \\ &(1-\eta)\cdot\left[\frac{1}{2}\cdot(1+z_2)\cdot\hat{c}_2(w\cdot(1+z_2)) - \frac{1}{2}\cdot z_2\cdot\hat{c}_2(w\cdot z_2) + \frac{1}{2}\cdot\hat{c}_2(w)\right].\end{aligned} \tag{24}$$

We now give our lower bound construction.

*Proof of Theorem 5.3.* Our proof has two cases, corresponding to the two cases of Lemma 5.1. First consider a set $\mathcal{C}$ and $\epsilon > 0$ so that the second case of the lemma applies. For a positive integer $m$, chosen later, we construct a game with player set $S$ and edge set $E$ — called resources here to avoid confusion with the tree described below — as follows.

1. Player strategies: Organize the resources in a tree of depth $m$, comprising a complete binary tree of depth $m-1$ with each leaf extended by a path of length 1. For each non-leaf node $i$ in the tree, there is a player $i$ with 2 strategies: either choose node $i$ or all children of $i$.
2. Player weights: If $i$ is the root, then $w_i = 1$; otherwise, if node $i$ is the left (right) child of some node $j$, then $w_i = w_j \cdot z_1$ ($w_i = w_j \cdot z_2$). Let $S_L$ be the set of players connected to a leaf.
3. Cost functions: Cost functions are defined recursively:

   (a) For the root, we pick any $c \in \mathcal{C}$ with $c(1) = 1$. Since $\mathcal{C}$ is closed under scaling and dilation, such a function exists.

   (b) Consider resource $e$ which is neither a leaf nor the root. Its cost function is $c_e$ and the weight of the corresponding player is $w_e$. Let $l, r$ be the left and right children of $e$, with corresponding player weights $w_l = z_1 \cdot w_e$ and $w_r = z_2 \cdot w_e$. Among all pairs of cost functions that satisfy (23) for $x^* = w_e$, pick a pair that also satisfies

$$c_j(w_e \cdot z_j) \cdot z_j = c_e(w_e) \tag{25}$$

   for $j \in \{1, 2\}$. Since $\mathcal{C}$ is closed under scaling and dilation, such a pair exists. Let $\eta_e$ be the corresponding value for $\eta$ in (24) and define

$$c_l(x) = \eta_e \cdot c_1(x) \quad \text{and} \quad c_r(x) = (1-\eta_e) \cdot c_2(x). \tag{26}$$

   (c) Every leaf resource gets the same cost function as its parent.
4. Nash strategies: The outcome $P$ where each player chooses the resource closer to the root.
5. Optimal strategies: The outcome $P^*$ where each player chooses the strategy further from the root.

We claim that the POA of the above game is $\lambda/(1-\mu)$, where $\lambda, \mu$ are the parameters in the second guarantee of Lemma 5.1. We first prove that $P$ is a PNE. It is clear that a player in $S_L$ has no incentive to deviate, since the leaf resource has the same cost as its current strategy. Consider a player $e \in S \setminus S_L$ and let $l, r$ be the left and right children respectively. Then, using (25) first and (24), (26) subsequently, we get

$$\begin{aligned}
\chi_{e,e}(\{e\}) &= w_e \cdot c_e(w_e) = w_e \cdot \eta_e \cdot z_1 \cdot c_1(w_e \cdot z_1) + w_e \cdot (1-\eta_e) \cdot z_2 \cdot c_2(w_e \cdot z_2) \\
&= w_e \cdot \left[\frac{1}{2} \cdot c_r(w_e) + \frac{1}{2} \cdot (1+z_2) \cdot c_r(w_e \cdot (1+z_2)) - \frac{1}{2} \cdot z_2 \cdot c_r(w_e \cdot z_2)\right] \\
&\quad + w_e \cdot \left[\frac{1}{2} \cdot c_l(w_e) + \frac{1}{2} \cdot (1+z_1) \cdot c_l(w_e \cdot (1+z_1)) - \frac{1}{2} \cdot z_1 \cdot c_l(w_e \cdot z_1)\right] \\
&= \chi_{e,r}(\{e,r\}) + \chi_{e,l}(\{e,l\}).
\end{aligned}$$

This completes the proof that $P$ is a PNE. Also, combining the second line of the above equality with (23), we get

$$\begin{aligned}
&\chi_{e,e}(\{e\}) = \lambda \cdot (\chi_{e,l}(\{e\}) + \chi_{e,r}(\{e\})) + \mu \cdot \chi_{e,e}(\{e\}) \quad \text{which gives} \\
&\chi_{e,e}(\{e\}) \geq (\zeta(\mathcal{C}) - \epsilon) \cdot (\chi_{e,l}(\{e\}) + \chi_{e,r}(\{e\})).
\end{aligned} \tag{27}$$

In the outcome $P$, the contribution of a non-leaf player $e$ to the total cost $C$ is equal to the combined contributions of the players corresponding to the left and right children $l$ and $r$ of $e$. This follows from (26) and (25):

$$z_1 \cdot w_e \cdot c_l(z_1 \cdot w_e) + z_2 \cdot w_e \cdot c_r(z_2 \cdot w_e) = z_1 \cdot w_e \cdot \eta_e \cdot c_1(z_1 \cdot w_e) + z_2 \cdot w_e \cdot (1-\eta_e) \cdot c_r(z_2 \cdot w_e) = w_e \cdot c_e(w_e).$$

It follows that, in $P$, the combined contribution of each layer of players in the tree is the same. Since the contribution of the root player is equal to 1, we get that $C = m$. Also,

$$\begin{aligned} C^* &= \sum_{i \in S} w_i \cdot \sum_{e \in P_i^*} \chi_{i,e}(\{i\}) \\ &= \sum_{i \in S_L} w_i \cdot \sum_{e \in P_i^*} \chi_{i,e}(\{i\}) + \sum_{i \in S \setminus S_L} w_i \cdot \sum_{e \in P_i^*} \chi_{i,e}(\{i\}) \\ &= \sum_{i \in S_L} w_i \chi_{i,i}(\{i\}) + \sum_{i \in S \setminus S_L} w_i \frac{\chi_{i,i}(\{i\})}{\zeta(\mathcal{C}) - \epsilon} = 1 + \frac{m-1}{\zeta(\mathcal{C}) - \epsilon}, \end{aligned}$$

where the last line follows from (27). The fact that $\lim_{m \to \infty} C/C^* = \zeta(\mathcal{C}) - \epsilon$ concludes the first case of the proof.

Finally, consider a set $\mathcal{C}$ and $\epsilon > 0$ so that the first guarantee of Lemma 5.1 applies. Thus, there is a triple $(c, x, x^*)$ with

$$\frac{1}{2} \cdot (x + x^*) \cdot c(x + x^*) - \frac{1}{2} \cdot x \cdot c(x) + \frac{1}{2} \cdot x^* \cdot c(x^*) \geq x \cdot c(x)$$

and

$$\frac{x \cdot c(x)}{x^* \cdot c(x^*)} \geq \zeta(\mathcal{C}) - \epsilon.$$

A similar construction yields the lower bound in this case. Let $z = x/x^*$. Then for $w > 0$ we have

$$\frac{1}{2} \cdot (z \cdot (w+1)) \cdot c(z \cdot (w+1)) - \frac{1}{2} \cdot z \cdot w \cdot c(z \cdot w) + \frac{1}{2} \cdot w \cdot c(w) \geq z \cdot w \cdot c(z \cdot w)$$

and

$$\frac{z \cdot c(z \cdot w)}{c(w)} \geq \zeta(\mathcal{C}) - \epsilon.$$

The resources are now organized on a path graph with a single player on each edge, who has to pick between the two endpoints. One end of the path is considered the root and has a cost function $c$ such that $c(1) = 1$. The weight of the adjacent player is 1. For all subsequent players, we multiply the weight by $z$, while each cost function $c_{i+1}$ is a dilated version of the previous one, satisfying $z \cdot c_{i+1}(z \cdot w) = c_i(w)$. The leaf node has the same cost function as its parent. The optimal profile has each player play further from the root, while in a PNE all players play closer to the root. The equilibrium condition holds because a deviation incurs a cost of $z \cdot c_{i+1}(z \cdot w + w)$, which is at least $z \cdot c_{i+1}(z \cdot w)$, which in turn equals $c_i(w)$. For all but the last player we get that the ratio of the cost in equilibrium to the cost in the optimal is

$$\frac{c_i(w)}{c_{i+1}(w)} = \frac{z \cdot c_{i+1}(z \cdot w)}{c_{i+1}(w)} \geq \zeta(\mathcal{C}) - \epsilon.$$

The last player has the same cost in both outcomes. The price of anarchy approaches $\zeta(\mathcal{C}) - \epsilon$ as the number of players grows to infinity, completing the proof. ☐

Remark 5.2. Here we illustrate this construction in the special case of cost functions that are polynomials with nonnegative coefficients and degree at most $d$. Note that for $c(x) = x^d$, and $x = \chi_d, x^* = 1$, the conditions for the first case of Lemma 5.1 hold (using the fact (20) that $3 \cdot \chi_d^{d+1} - 1 = (\chi_d + 1)^{d+1}$).

We can therefore apply the path construction from the second part of the proof of Theorem 5.3 with $z = \chi_d$. The weight of the $i$th player is $\chi_d^{i-1}$ and the cost function of the $j$th resource is $\chi_d^{(1-j)\cdot(d+1)} \cdot x^d$ — except for the last one, which has the same cost function as its neighbor. The POA in this example is $\chi_d^{d+1}$, matching the upper bound in Theorem 5.2.

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