Bonus Lecture #2: Digital Signatures in Blockchain Protocols (Part 1 of 2)

COMS 4995-001:
The Science of Blockchains

URL: https://timroughgarden.org/s25/

Tim Roughgarden

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Goals for Bonus Lecture #2

1. Bits of security.
   – what does it mean and how much is enough?
2. Groups and the discrete logarithm approach to signatures.
   – common to ECDSA, Schnorr, BLS, etc.
3. Algorithms for the discrete logarithm problem.
   – the discrete log problem is not as hard as you might have thought!

Digital Signature Schemes in Blockchains

- one of the two most ubiquitous cryptographic primitives used in blockchain protocols (along with cryptographic hash functions)

Application #1: allows a user of a blockchain to authorize a transaction (e.g., making a payment).

- fundamental to the vision of shared computer in the sky

Application #2: under the hood, allows validators of a blockchain protocol to sign their messages.

- used in most blockchain protocols for this purpose

Defining Digital Signature Schemes

Digital signature scheme: defined by 3 (efficient) algorithms:

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   – in some cases, may generate r itself (e.g., ssh-keygen)
2. *Signing algorithm:* maps message + sk → signature.
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  - if there's an attack faster than brute force ➔ < t bits of security

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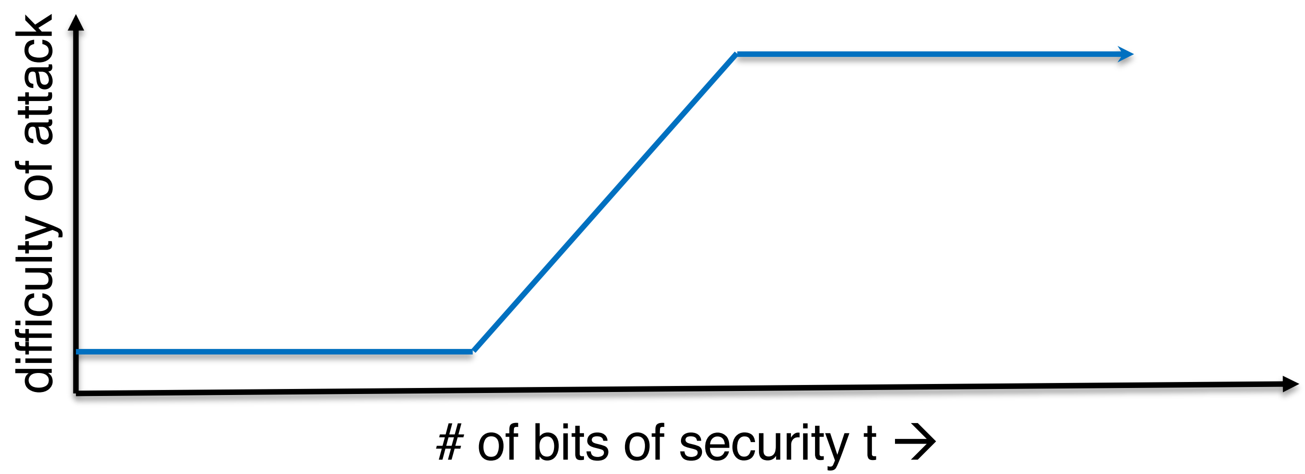

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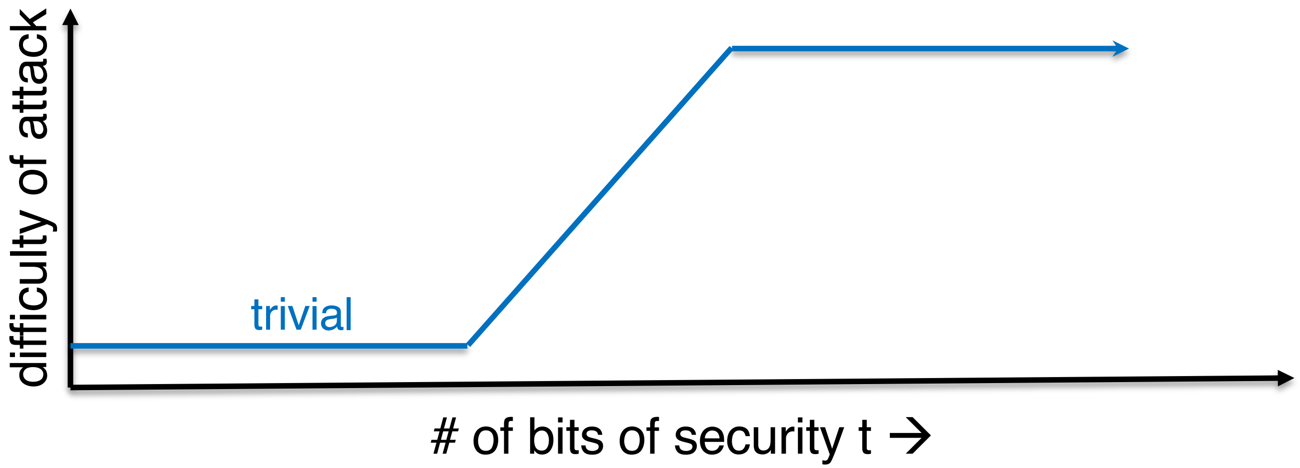

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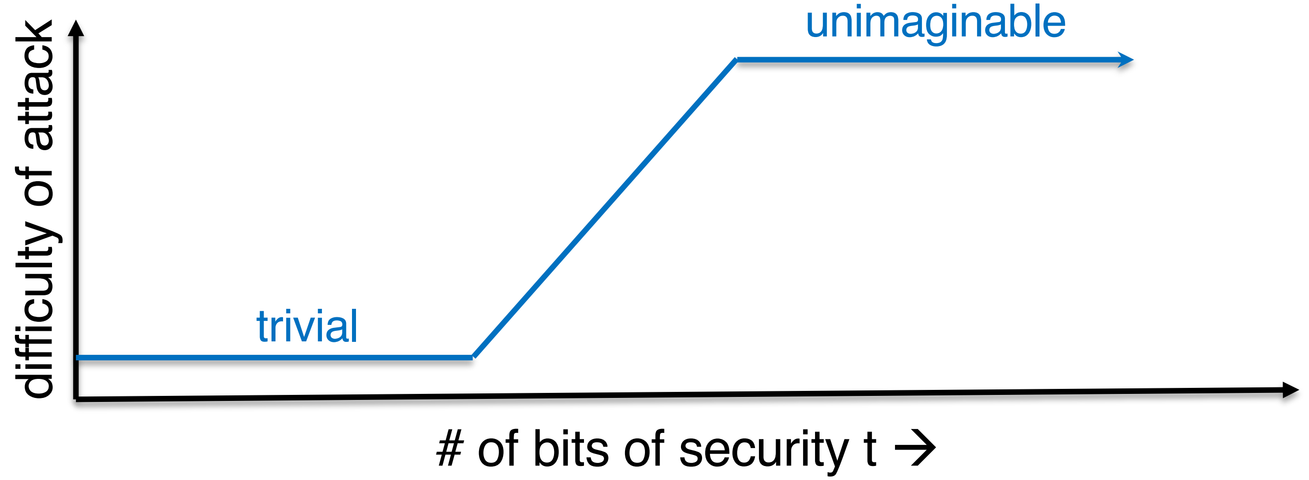

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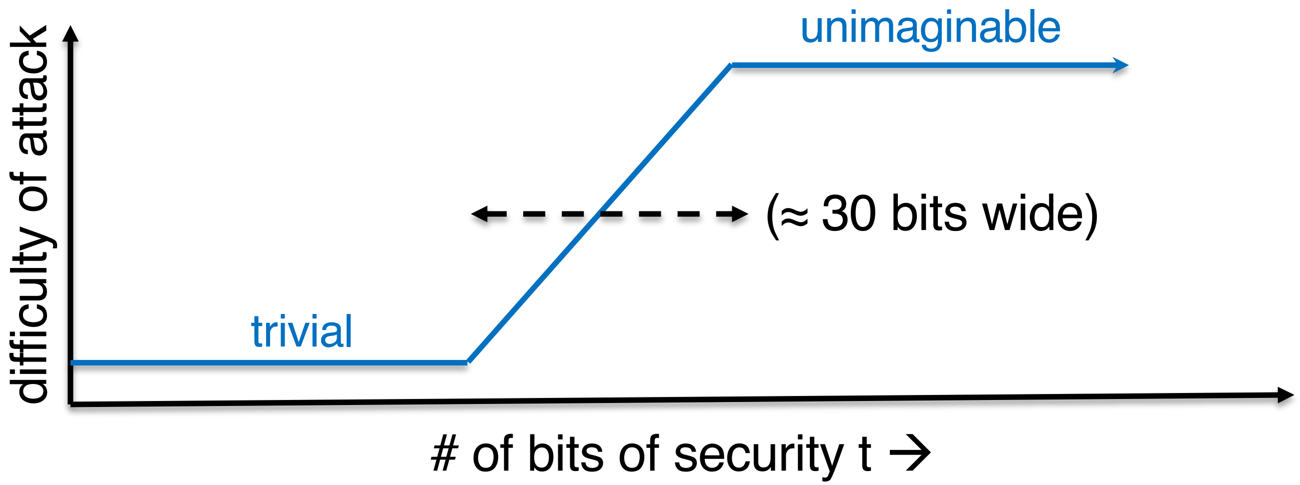

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Upshot:

- 80 bits of security fine 20 years ago, not good enough now
- 128 bits regarded as plenty in the short- and medium-term

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  - issue: no hope of proving this (would imply P ≠ NP!)

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- note: can similarly invert f with O(t) multiplications
  - repeatedly square, overshoot the target, divide out, repeat

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- a chosen as a "generator of Z_p^*" [explained next]

Generators of Cyclic Groups

Notation: $Z_p^* = \{1,2,\ldots,p\text{-}1\}$.

Example: 3 is a generator of $Z_7^* = \{1,2,\ldots,6\}$:

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- pk := g^{sk} (can be computed efficiently by repeated squaring)
  - conceptually, multiply (i.e., group operation) g by itself sk times

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Fact: if the group G has order q, can solve the discrete log problem for G with $O(\sqrt{q})$ group operations.

Consequence: with DL approach, need key size ≥ 256 bits to get 128 bits of security [no matter what G is].

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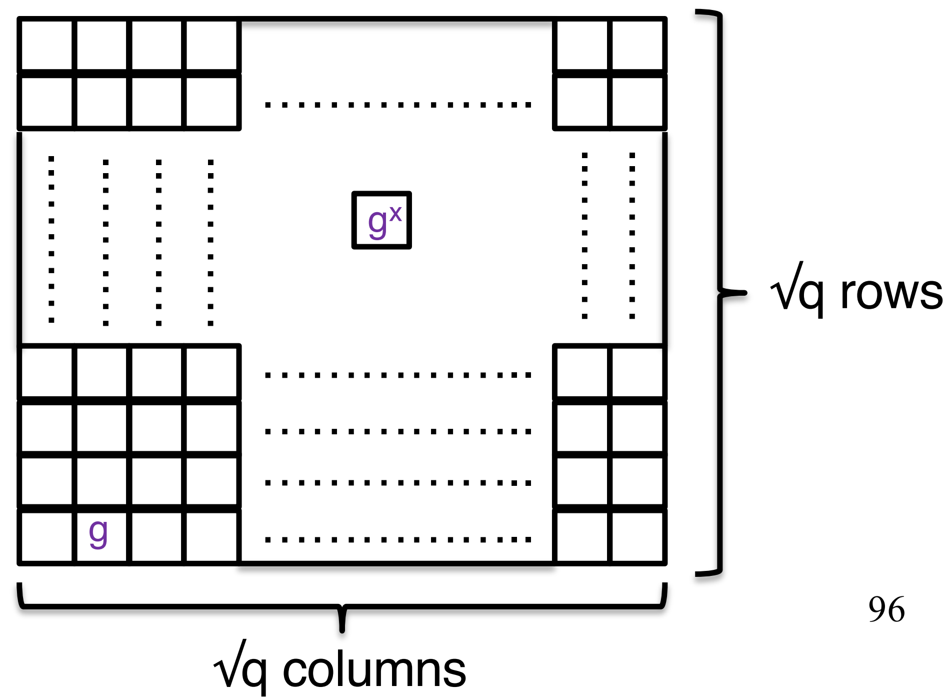

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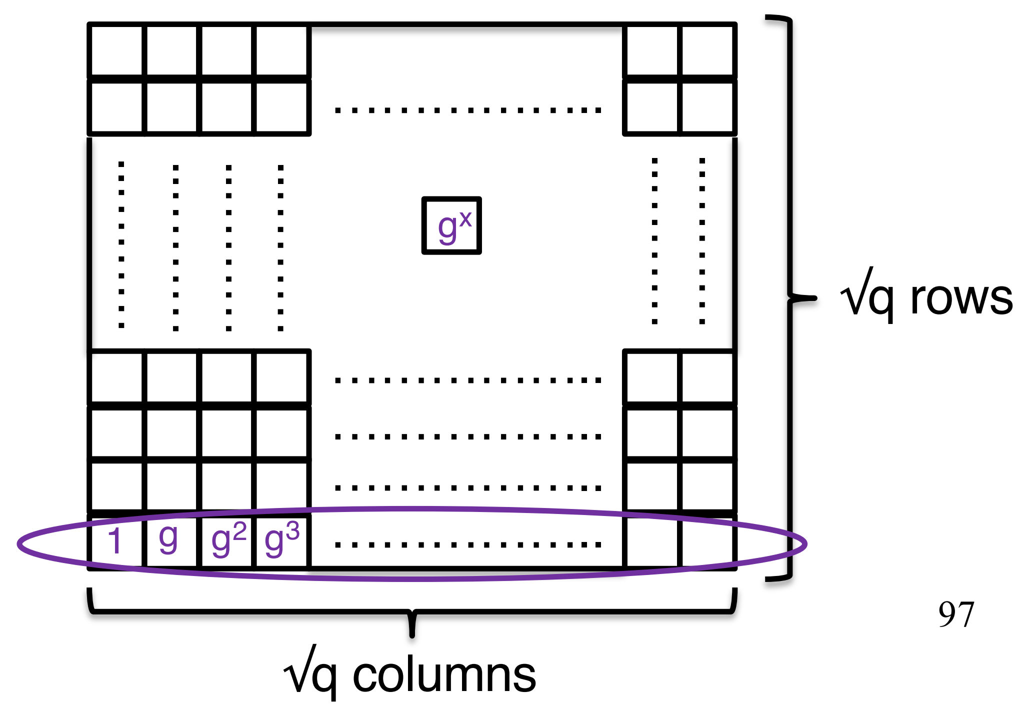

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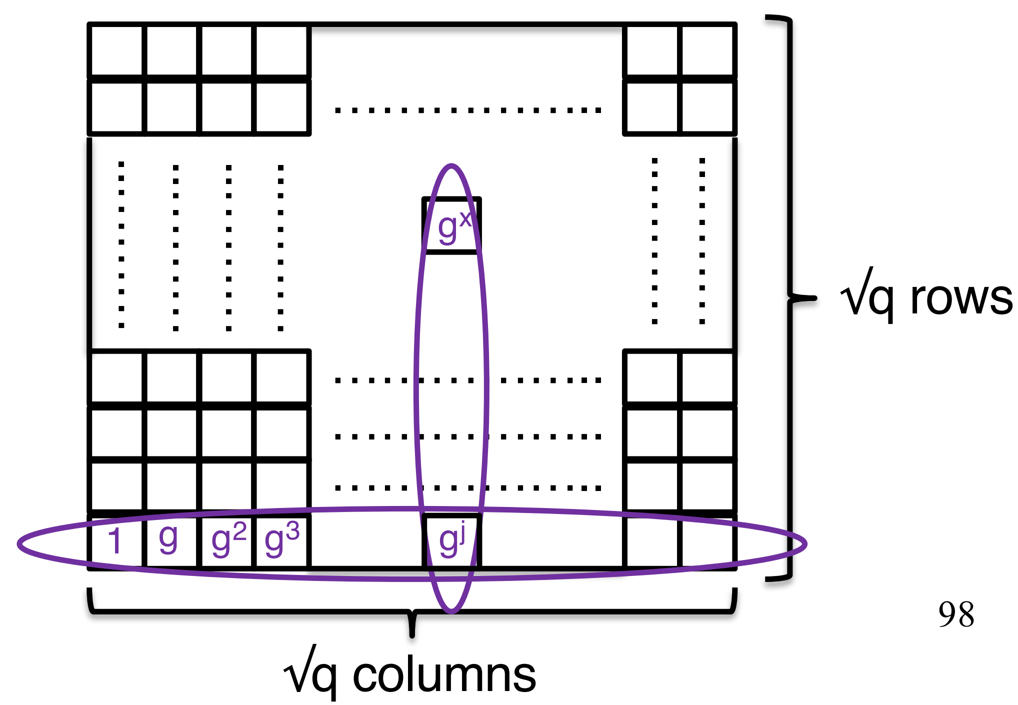

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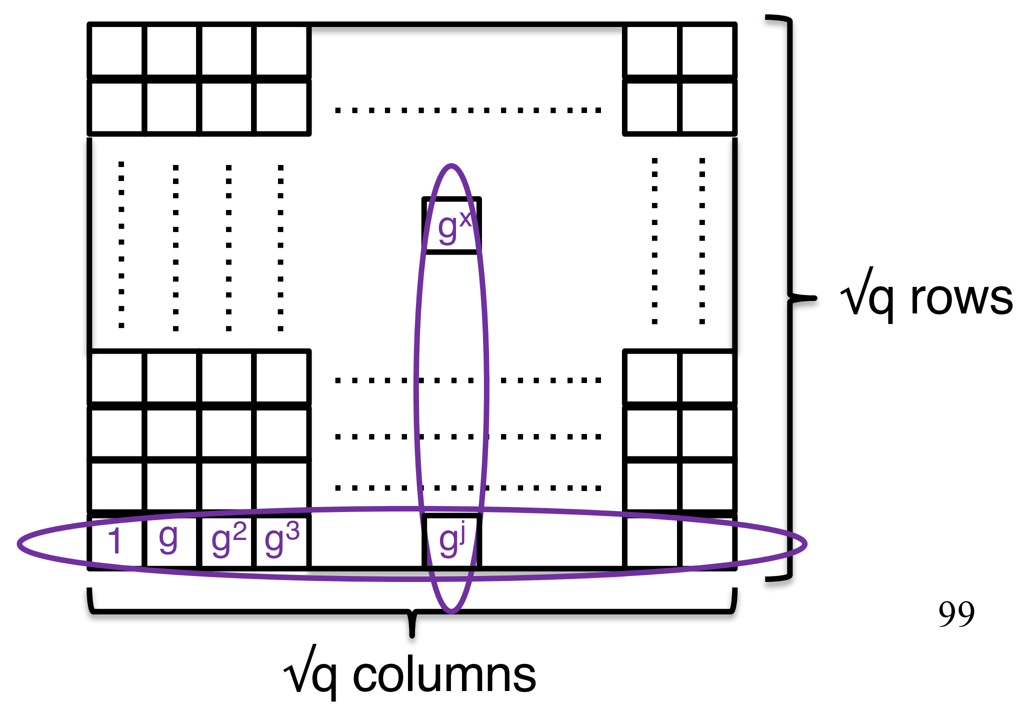

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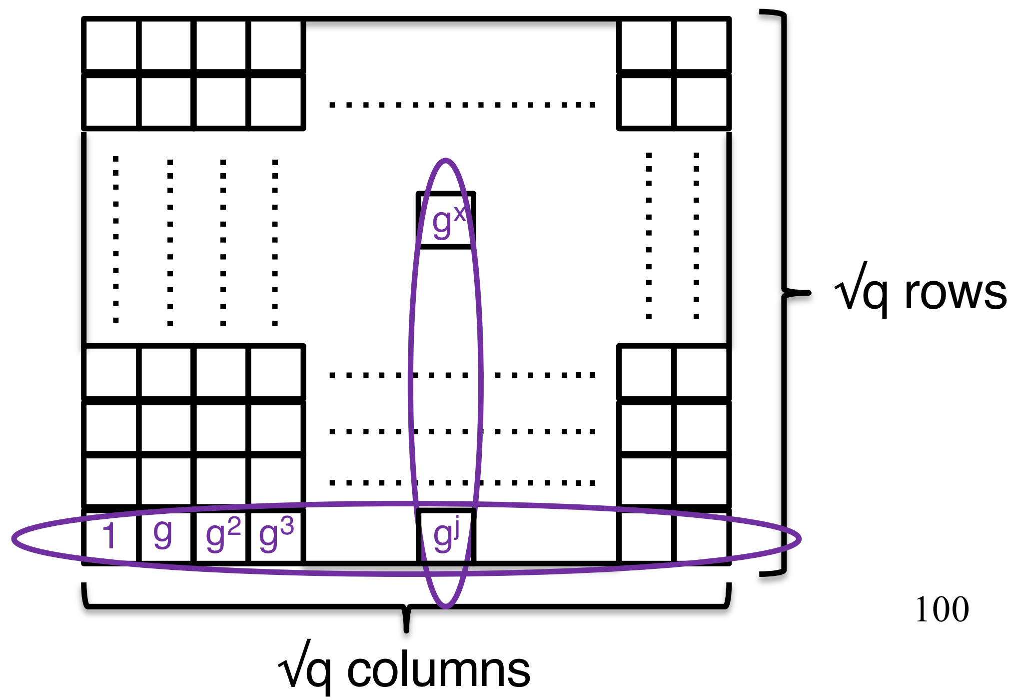
➔ uses at most $2\sqrt{q}$ group operations!



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Fact: “black-box/generic” algorithms cannot solve the discrete log problem with $o(\sqrt{q})$ group operations [where q = order of group].

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- all blockchain protocols will likely need to upgrade to post-quantum-secure signature schemes in the next decade or two
  - arguably, less urgent than for e.g. encryption of sensitive data
  - likely to cause a non-trivial performance hit