Bonus Lecture #4:
KZG Commitments

COMS 4995-001:
The Science of Blockchains

URL: https://timroughgarden.org/s25/

Tim Roughgarden

Goals for Bonus Lecture #4

1. Fun facts about polynomials.
   – roots of polynomials, encoding data with a polynomial
2. KZG commitments: the basic idea.
   – commitment, proofs via polynomial evaluation
3. Making it real: structured reference string + group pairings.
   – implementing the idea of “evaluation at an unknown random input”
4. Trusted setup ceremonies.
   – where does the structured reference string come from?

Roots of Polynomials

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Fact 2: if polynomial f satisfies $f(r) = 0$, then f has the form $f(x) = (x - r) \cdot q(x)$, where q is a degree-(d-1) polynomial.

- example: can write $x^3 - 6x^2 + 11x - 6$ as $(x - 1)(x^2 - 5x + 6)$
- for proof(s), see “polynomial factor theorem” (e.g., use long division)

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– for randomly chosen x (from big set), p(x),q(x) almost certainly differ

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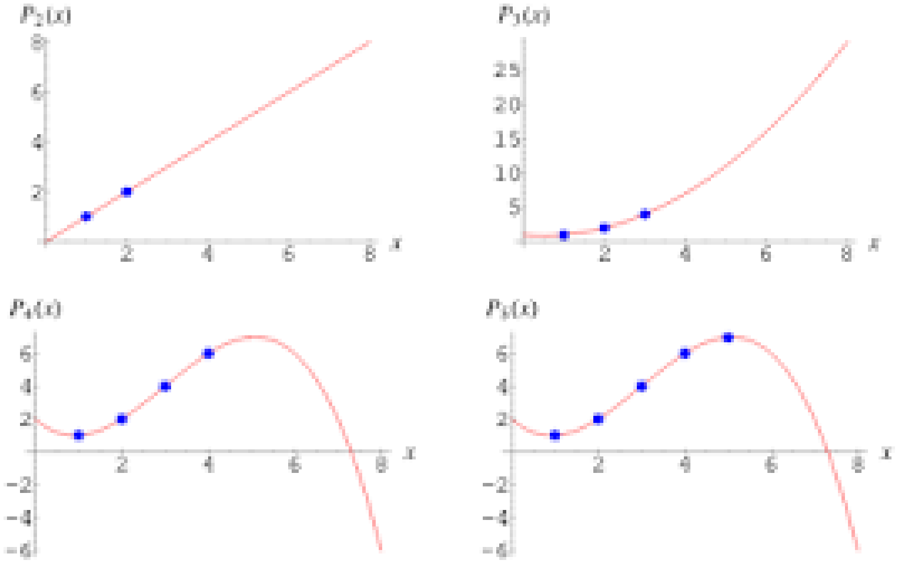

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– also: can redundantly encode polynomial via > d+1 evaluations

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Example: list $a_0, a_1, \ldots, a_d$ of objects, each a_i with length 256 bits.

- interpret as elements $y_0, y_1, \ldots, y_d$ of Z_p={0,1,…,p-1} for ≈256-bit prime p
- can compute degree-d polynomial f s.t. $f(i)=y_i$ for all i=0,1,2,…,d
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Ethereum blob: as above, with d = 4096 (size ≈ 125 Kb).

– validators store blobs for 2 weeks, (KZG) commitments to blobs forever

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(ii) [intuition] if $f(z) \neq v$ ➔ for any w, $(x - z) \cdot w(x) \neq f(x) - v$ ➔ for almost all τ, $(\tau - z) \cdot w(\tau) \neq f(\tau) - v$ [Corollary 2]

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Question: how to compute $f(\tau)$ without knowing τ?

Idea: compute $g^{f(\tau)}$, where g = generator of some cyclic group.

- e.g., of an elliptic curve group
- i.e., compute $f(\tau)$ only in "encrypted form" (i.e., in exponent)
- cf., Schnorr signatures

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Tl;dr: easy to add or scale in the exponent, but hard to multiply.

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- issue: easy to add/scale in exponent, but hard to multiply
- solution: assume "powers of tau" $\sigma = (g^{\tau}, g^{\tau^2}, g^{\tau^2}, \ldots, g^{\tau^d})$ are known (e.g., part of description of commitment scheme)
  - σ called a "structured reference string," a form of "trusted setup"

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Fact: there exist groups ("elliptic curve groups with pairings") in which can efficiently verify multiplication in the exponent.

- given input $x = g^{a}$, $y = g^{b}$, $z = g^{c}$, reports whether $c = a \cdot b$

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[Kate/Zaverucha/Goldberg 2010]

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Guarantee: ≥ 1 honest participant (i.e., chooses its τ_i randomly and deletes it forever) ➔ τ is effectively random and unknown!