CS364A: Problem Set #1

Due in class on Thursday, October 19, 2006

Instructions:

(1) This is a challenging problem set, and you are not expected to solve all of the problems to completion. You are, however, expected to think hard about all of them. Give complete solutions for as many as you can; for the others, explain your progress and where you got stuck. (If you've spent, say, 12 hours on the problem set and are sick of it, then you should just turn it what you have.)

(2) You may refer to your course notes, general references (e.g., textbooks), and material on the course Web site, but *not* to specific research papers.

(3) Collaboration on this homework is actively encouraged. However, your write-up must be your own, and you must list the names of your collaborators on the front page.

(4) Grades will be assigned on a plus/check/minus scale (standing for exceptional, satisfactory, and a bit lacking, respectively).

Problem 1

Recall from class that a digital goods auction is one in which there are n buyers (each with a private valuation) interested in one copy of a good, and n copies of this good. Recall that in a threshold auction, for each bidder i a price $p_i = t_i(b_{-i})$ is computed based on the bids b_{-i} of the other players, and bidder i is given the good and charged price p_i if $p_i < b_i$ and is not given the good if $p_i > b_i$. (Throughout this problem, ignore the issue of how to break ties when $p_i = b_i$.)

(a) Call an auction *monotone* if it has the following property for every bidder i and every fixed set b_{-i} of bids by the other players: if bidder i wins the good when bidding b_i and $b'_i \geq b_i$, then bidder i also wins the good when bidding b'_i.

Prove that every truthful, individually rational (IR) digital goods auction is monotone.

(b) Fix a bidder i and a set b_{-i} of bids by the other players. Fix a truthful, IR auction. Prove that if bidder i wins the good with two different bids b_i and b'_i, then it is charged the same price in both cases.

(c) Fix a bidder i and a set b_{-i} of bids by the other players. Fix a truthful, IR auction. Let p_i denote the price charged to bidder i when it wins the good; note p_i is well defined by part (b). Prove that bidder i must win when its bid b_i is greater than p_i and must lose when $b_i < p_i$.

(d) Prove that every truthful, IR digital goods auction is equivalent to a threshold auction.

Problem 2

Recall from class that, for digital goods auctions, we always assumed that there were n copies of the good. In this problem you will show that the case where there are only $k < n$ copies of the good available reduces to the case where $k = n$.

Recall that an auction is *c-competitive with respect to* $\mathcal{F}^{(2)}$ if, for every bid vector b, its expected revenue is at least a $1/c$ fraction of $\mathcal{F}^{(2)}(b) = \max_{2 \leq i \leq n} i \cdot b_i$. (As usual, we assume without loss that $b_1 \geq b_2 \geq \cdots \geq b_n$.)

Suppose that, for some $c \geq 1$, there is a truthful auction for the $k = n$ case that is c-competitive with respect to $\mathcal{F}^{(2)}$. Fix $k \in \{2, 3, \ldots, n\}$. Show that there is a truthful auction for the case with k copies of the good that is c-competitive with respect to $\mathcal{F}^{(2,k)}$, where $\mathcal{F}^{(2,k)}$ is the maximum-possible revenue via a fixed price that sells to at least 2 and at most k bidders:

$$\mathcal{F}^{(2,k)}(b) = \max_{2 \leq i \leq k} i \cdot b_i.$$

Problem 3

All of the auctions that we'll study in class are *offline*, in the sense that all of the bidders are present at the beginning of the auction. In this problem we'll consider *online* auctions, where bidders arrive one by one. Specifically, the model is this: n bidders arrive one at a time; when a bidder shows up, it presents a nonnegative bid; the auction decides whether or not to sell a copy of a good to the bidder (assuming there are still goods to be sold), and at what price (constrained above by the bid); then the bidder departs, never to return.

(a) For starters we consider a non-game-theoretic setting, and we assume without justification that bidders bid their true valuations. Also, suppose you have only *one* copy of a good (like in the Vickrey auction). Thus the maximum possible surplus is simply the highest valuation; since bidders are assumed to bid truthfully, this is also the maximum possible revenue (once you decide who to sell to, you can just charge them their bid).

Prove that if the valuations of bidders and the order in which they arrive are arbitrary, then no constant-factor approximation of the maximum revenue is possible. (Here, an auction c-approximates the revenue for $c \geq 1$ if it always obtains revenue that is at least $1/c$ times the maximum possible.)

(b) To obtain a tractable problem, we now assume that bidders arrive in a *random* order. Precisely, assume that the number n of bidders is publicly known and that bidders bid truthfully. An adversary chooses n arbitrary nonnegative valuations, and then the bidders are ordered uniformly at random.

Design an online algorithm that, in expectation over the random ordering of the bidders, c-approximates the revenue for some constant $c > 1$. Obtain the smallest constant that you can.

(c) Remove the assumption that bidders automatically bid their true valuations by turning your algorithm from part (b) into a truthful online auction. Include a brief proof of truthfulness.

(d) **Extra Credit:** Can you prove a matching negative result (a lower bound on the smallest constant achievable by any online algorithm) that matches your bound in part (b)?

(e) **Extra Credit:** What can you say about the generalization of the problem in which you have $k > 1$ copies of a good to sell, and would like to maximize expected revenue? In particular, is the best-possible approximation factor larger or smaller than your answers for (b,d) for other values of k (e.g., $k = 2$).

Problem 4

Recall from class the *winner determination (WD)* problem for a combinatorial auction: given the valuations $v_1, \ldots, v_n$ (each a function from 2^S to the nonnegative reals, where S is the set of goods), compute an allocation $T_1^*, \ldots, T_n^*$ that maximizes the surplus $\sum_{i=1}^{n} v_i(T_i)$ over all feasible allocations $\{T_i\}$. Recall that in the WD problem, we ignore all incentive constraints and assume that these valuations are known to the algorithm.

The WD problem is difficult, even when the valuations have special structure. In this problem we will assume that each valuation v_i satisfies the following properties:

(A1) for every subset $T \subseteq S$ of goods, the value $v_i(T)$ can be computed in polynomial time (i.e, in time polynomial in the number m of goods);

(A2) v_i is nondecreasing: for every $T_1 \subseteq T_2 \subseteq S$, $v_i(T_1) \leq v_i(T_2)$;

(A3) v_i is *submodular*, meaning that for every $T_1 \subseteq T_2 \subseteq S$ and every $j \notin T_2$,

$$v_i(T_2 \cup \{j\}) - v_i(T_2) \leq v_i(T_1 \cup \{j\}) - v_i(T_1).$$

(This is one way of expressing "diminishing returns".)

(a) (Easy) Let $A, B \subseteq S$ be two disjoint subsets of goods. For $T \subseteq B$, define $v_i^A(T)$ as $v_i(A \cup T) - v_i(T)$. Prove that submodularity of v_i on S implies submodularity of v_i^A on B.

(b) Consider the following heuristic for the WD problem with valuations satisfying (A1)–(A3): order the goods $1, 2, \ldots, m$ arbitrarily; allocate the goods one-by-one in this order, giving the good j to the player whose valuation would increase the most. In other words, if the players possess the bundles $T_1, \ldots, T_n$ after the first $j - 1$ goods have been allocated, then award the jth good to the player with maximum $v_i(T_i \cup \{j\}) - v_i(T_i)$.

Determine the smallest constant α for which the following is true: the output of this heuristic always has surplus at least $1/\alpha$ times the maximum possible. We are interested in both upper and lower bounds on this constant.

(For the upper bound, you might find (a) useful in conjunction with an inductive argument. But if you can prove an upper bound without explicitly using (a), that's fine too.)

(c) **Extra Credit:** Let's revisit assumption (A1), which states that for every player i and bundle $T \subseteq S$, we can compute $v_i(T)$ in time polynomial in $m = |S|$. Explicitly storing all possible values of $v_i(T)$ requires a look-up table with $2^m - 1$ entries—one for each non-empty value of T. For this reason, such a valuation is typically represented "implicitly". For example, perhaps player i has a value $v_{ij} \geq 0$ for each good $j \in S$, and its value for a bundle is simply the sum of these values: $v_i(T) = \sum_{j \in T} v_{ij}$. This is called an *additive* valuation. Note that $v_i(T)$ might take on 2^m distinct values (e.g., if the v_{ij}'s are distinct powers of 2), yet it can be completely described using m numbers and a given value $v_i(T)$ can be computed in $O(m)$ time. Also note that an additive valuation is both nondecreasing and submodular.

Give an example of such an "implicit description" of a nondecreasing valuation (where $v_i(T)$ is some function of a polynomial (in m) number of parameters) such that computing $v_i(T)$ is NP-hard. Can you extend your hardness result to apply to implicitly described submodular valuations?

(d) **Extra Credit:** Give a family of implicitly described (in the sense of (c)) valuations that satisfy assumptions (A1)–(A3) and such that the winner determination problem for such valuations is NP-hard. (Hint: modify additive valuations in a way that preserves submodularity but makes surplus maximization hard.)