CS364A: Exercise Set #7

Due by the beginning of class on Wednesday, November 13, 2013

Instructions:

(1) Turn in your solutions to all of the following exercises directly to one of the TAs (Kostas or Okke). Please type your solutions if possible and feel free to use the LaTeX template provided on the course home page. Email your solutions to `cs364a-aut1314-submissions@cs.stanford.edu`. If you prefer to hand-write your solutions, you can give it to one of the TAs in person at the start of the lecture.

(2) Your solutions will be graded on a "check/minus/zero" system, with "check" meaning satisfactory and "minus" meaning needs improvement.

(3) Solve these exercises and write up your solutions on your own. You may, however, discuss the exercises verbally at a high level with other students. You may also consult any books, papers, and Internet resources that you wish. And of course, you are encouraged to contact the course staff (via Piazza or office hours) to clarify the questions and the course material.

(4) No late assignments will be accepted.

Lecture 13 Exercises

Exercise 55

Consider an atomic selfish routing game with affine cost functions. Let $C(f)$ denote the total travel time of a flow f and $\Phi(f)$ the value of Rosenthal's potential function for f. Prove that

$$\frac{1}{2}C(f) \leq \Phi(f) \leq C(f)$$

for every flow f.

Exercise 56

Algorithmic Game Theory, Exercise 18.4. Note this exercise refers to atomic selfish routing games with *weighted* players, where different players can control different amounts of flow. Example 18.7 from the AGT book shows that with quadratic cost functions, pure Nash equilibria need not exist in such routing games. (But this exercise asks about affine cost functions.)

Exercise 57

In lecture, we defined a mixed Nash equilibrium of a cost-minimization game to be a set $\sigma_1, \ldots, \sigma_k$ of distributions over the strategy sets $A_1, \ldots, A_k$ such that

$$\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(s_i', \mathbf{s}_{-i})]$$

for every player i and (pure) deviation $s_i' \in A_i$, where $\sigma = \sigma_1 \times \cdots \times \sigma_k$ is the product distribution induced by the players' mixed strategies.

Suppose instead we allow mixed-strategy deviations. That is, consider the distributions $\sigma_1, \ldots, \sigma_k$ that satisfy

$$\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{s_i'\in\sigma_i', \mathbf{s}\sim\sigma}[C_i(s_i', \mathbf{s}_{-i})]$$

for every player i and distribution σ_i' over A_i. Show that $\sigma_1, \ldots, \sigma_k$ satisfy this condition if and only if it is a mixed Nash equilibrium in the sense of our original definition.

Exercise 58

Consider a cost-minimization game and a product distribution $\sigma = \sigma_1 \times \cdots \times \sigma_k$. Show that σ is a correlated equilibrium of the game if and only if $\sigma_1, \ldots, \sigma_k$ form a mixed Nash equilibrium of the game.

Exercise 59

Consider a cost-minimization game. Prove that a distribution σ over outcomes $A_1 \times \cdots \times A_k$ is a correlated equilibrium if and only if it has the following property: for every player i and function $\delta : A_i \to A_i$,

$$\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\delta(s_i), \mathbf{s}_{-i})] \, .$$

Exercise 60

Prove that every correlated equilibrium of a cost-minimization game is also a coarse correlated equilibrium.

Exercise 61

Consider an atomic selfish routing network that has four players with the same source s and destination t, and six parallel edges from s to t, each with cost function $c(x) = x$.

Consider the distribution σ over outcomes that randomizes uniformly over all outcomes with the following properties:

(1) There is one edge with two players.

(2) There are two edges with one player each (so three edges are empty).

(3) The set of edges with at least one player is either $\{1, 3, 5\}$ or $\{2, 4, 6\}$.

Prove that σ is a coarse correlated equilibrium but not a correlated equilibrium.

Lecture 14 Exercises

Exercise 62

Prove that there is a location game in which the POA of pure Nash equilibria is $\frac{1}{2}$, matching the worst-case bound given in lecture.

Exercise 63

Prove that every location game is a potential game in the sense of Lecture 13. What is the potential function?

Exercise 64

Prove that if $\mathbf{s}$ is an ϵ-approximate Nash equilibrium of a (λ, μ)-smooth cost-minimization game — meaning that $C_i(\mathbf{s}) \leq (1 + \epsilon)C_i(s'_i, \mathbf{s}_{-i})$ for every player i and deviation $s'_i \in A_i$ — with $\epsilon < \frac{1}{\mu} - 1$, then the cost of $\mathbf{s}$ is at most $\frac{\lambda(1+\epsilon)}{1-\mu(1+\epsilon)}$ times that of an optimal outcomes.