CS269I: Exercise Set #3

Due by 11:59 PM on Wednesday, October 17, 2018

Instructions:

(1) You can work individually or in a pair. If you work in a pair, the two of you should submit a single write-up.

(2) Submission instructions: We are using Gradescope for the homework submissions. Go to www.gradescope.com to either login or create a new account. Use the course code MZZ2BV to register for CS269I. Only one person needs to submit the assignment. When submitting, please remember to add your partner's name (if any) in Gradescope.

(3) Please type your solutions if possible. We encourage you to use the LaTeX template provided on the course home page.

(4) Write convincingly but not excessively. You should be able to fit all of your solutions into two pages, if not less.

(5) Except where otherwise noted, you may refer to the course lecture notes and the specific supplementary readings listed on the course Web page *only*.

(6) You can discuss the exercises verbally at a high level with other groups. And of course, you are encouraged to contact the course staff (via Piazza or office hours) for additional help.

(7) If you discuss solution approaches with anyone outside of your group, you must list their names on the front page of your write-up.

(8) No late assignments will be accepted, but we will drop your lowest exercise set score.

Lecture 5 Exercises

Exercise 11

Recall the First Welfare Theorem for the model covered in lecture (with $r_j = 0$ for every good j): if $(M, \mathbf{p})$ is a competitive equilibrium (where M is a matching and $\mathbf{p}$ is a price vector indexed by the goods), then

$$\sum_{i=1}^{n} v_{iM(i)} \geq \sum_{i=1}^{n} v_{iM^*(i)}$$

for every matching M^*. (Reminders: $M(i)$ denotes the good assigned to i in M or its outside option, as appropriate; outside options have value 0; and v_{ij} denotes the valuation of buyer i for good j.)

Use the First Welfare Theorem to prove that every competitive equilibrium $(M, \mathbf{p})$ is a Pareto optimal outcome. That is, prove that for every other matching M' and price vector $\mathbf{q}$, if some buyer or seller is strictly better off in $(M', \mathbf{q})$ than in $(M, \mathbf{p})$, then some other buyer or seller is strictly worse off in $(M', \mathbf{q})$ than in $(M, \mathbf{p})$. (By definition, a buyer i is better/worse off if $v_{iM'(i)} - q_{M'(i)}$ is bigger/smaller than $v_{iM(i)} - p_{M(i)}$; the seller of a good j is better/worse off if q_j is bigger/smaller than p_j.)

Exercise 12

Now suppose that the seller of a good j is allowed to have an arbitrary nonnegative reserve price r_j.[1]

(a) Redefine a competitive equilibrium for this more general setting.

(b) Prove an analog of the First Welfare Theorem for this more general setting.

Lecture 6 Exercises

Exercise 13

In lecture, we briefly mentioned the idea of "synergies" between goods when discussing the exposure problem. Informally, there are synergies between goods if you have significant value for a subset of goods only in the presence of one or more additional goods.

To make this precise, suppose that each buyer has a valuation (i.e., maximum willingness to pay) $v_i(S)$ for each subset S of the goods.[2] We say that the function v_i (from subsets of goods to the nonnegative real numbers) is *subadditive* if for every pair A, B of disjoint bundles of goods, $v_i(A \cup B) \leq v_i(A) + v_i(B)$.

(a) Interpret the formal definition of subadditivity in terms of our informal notion of synergies between goods.

(b) In our examples at the end of lecture illustrating demand reduction and the exposure problem, we considered three different valuation functions.[3] Which of these three functions, if any, are subadditive? Justify your answer.

[1] For part (b), you might want to also interpret r_j as a seller's valuation for its own good.

[2] In Lecture 5 buyers were assigned at most one good, so we needed only one valuation per buyer-good pair; in a spectrum auction, a buyer might buy any subset of licenses, so we need a valuation for each subset.

[3] In the first example, the first bidder was willing to pay 6 for Northern California, 6 for Southern California, and 12 for both. In the second example, the first bidder was willing to pay 6 for the pair of Northern and Southern California, and nothing for anything less. In both examples, the second bidder only wanted one license, and was willing to pay 5 for either one.