COMS 4995 (Randomized Algorithms): Exercise Set #2

For the week of September 9–13, 2019

Instructions:

(1) *Do not turn anything in.*

(2) The course staff is happy to discuss the solutions of these exercises with you in office hours or in the course discussion forum.

(3) While these exercises are certainly not trivial, you should be able to complete them on your own (perhaps after consulting with the course staff or a friend for hints).

Exercise 4

Let X, Y be independent random variables on the same probability space. Prove that

$$\mathbf{E}[X \cdot Y] = \mathbf{E}[X] \cdot \mathbf{E}[Y] \,.$$

Exercise 5

Give an example of three random events X, Y, Z for which any pair are independent but all three are not mutually independent.

Exercise 6

Recall from Lecture #2 that an instance of MAX 3SAT is specified by n Boolean variables $x_1, x_2, \ldots, x_n$ and m clauses, where each clause is the OR of three distinct literals (and where a literal is either a variable or its negation), such as $\neg x_1 \vee x_2 \vee \neg x_3$. Recall also that a random truth assignment (chosen uniform at random from the 2^n possibilities) satisfies, in expectation, $\frac{7}{8}m$ of the clauses.

(a) Prove that if there is at least one truth assignment that satisfies strictly less than 87.5% of the clauses, then there is also a truth assignment that satisfies strictly more than 87.5% of the clauses.

(b) Exhibit an instance of MAX 3SAT for which *every* truth assignment satisfies exactly 87.5% of the clauses.

Exercise 7

Consider n variables, each of which can be assigned any value in $\{1, 2, \ldots, q\}$.[1] Consider m constraints, where a constraint consists of a subset of distinct variables and a collection of tuples of assignments to these variables that satisfy the constraint. (In 3SAT, every constraint can be regarded as a list of the 7 3-tuples of truth assignments to its variables that satisfy it.) This is an instance of a *constraint satisfaction problem (CSP)*.

Now suppose that every constraint involves exactly r variables[2] and lists exactly ℓ tuples. (E.g., in MAX 3SAT, $r = 3$ and $\ell = 7$.) What is the expected number of constraints satisfied by a uniformly random assignment (as a function of n, q, m, r, and ℓ)?

[1] The Boolean case corresponds to $q = 2$. The parameter q is sometimes called the *alphabet size*.

[2] Called an *arity-r CSP* or simply *r-CSP*.

Exercise 8

Recall from Lecture #2 that the .878-approximation guarantee for the Goemans-Williamson randomized hyperplane rounding algorithm boiled down to the following assertion:

$$\frac{\theta}{\pi} \geq .878 \cdot \tfrac{1}{2}(1 - \cos\theta)$$

for all $\theta \in [0, \pi]$ or, equivalently, that

$$\min_{\theta \in [0,\pi]} \frac{2\theta}{\pi(1 - \cos\theta)} \geq .878.$$

Use your favorite plotting program (e.g., Matlab, Mathematica, GNUplot, etc.) to convince yourself that this assertion is true.

Exercise 9

Recall from Lecture #3 that an n-vertex undirected graph has at most $\binom{n}{2}$ minimum cuts.

Given an undirected graph $G = (V, E)$ and two distinct vertices $s, t \in V$, an *s-t cut* of G is a partition (A, B) of the vertex set V with $s \in A$ and $t \in B$. The *value* of a cut is defined as before, as the number of edges with one endpoint in each of A and B.

Prove that the maximum number of minimum s-t cuts of an n-vertex undirected graph grows exponentially with n.

[Hint: Use $2(n-2)$ edges, with a minimum s-t cut value of $n-2$.]

Exercise 10

In a directed graph $G = (V, E)$, a *cut* is defined as an ordered partition of the vertex set V into two non-empty sets, the first set A and the second set B. The *value* of such a cut (A, B) is defined as the number of edges directed from A to B (i.e., edges going backward across the cut from B to A are not counted).

Prove that the maximum number of minimum cuts of an n-vertex directed graph grows exponentially with n.

[Hint: Start from the previous construction, and add many parallel arcs backward to force all minimum cuts to be s-t cuts.]