Uncoupled Potentials for Proportional Allocation Markets

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***Abstract*— We study resource allocation games where allocations to agents are made in proportion to their bids. We show that the existence of a potential function in the allocation space, and a virtual price function are sufficient for the convergence of better response dynamics to Nash equilibrium. Generally, resource allocation games do not admit a potential in their strategy space, and are not in the class of potential games. However, for many interesting examples, including the Kelly mechanism, the best response functions are "well-behaved" on the allocation space, and consequently a potential in that space exists.**

We demonstrate how our sufficient condition is satisfied by three classes of market mechanisms. The first is the class of smooth market-clearing mechanisms, where the market is cleared using a single nondiscriminatory price. The second example is the class of simple g-mechanisms where an efficient Nash equilibrium is implemented with price discrimination. Finally we show our results apply to a subset of scalar strategy VCG (SSVCG) mechanisms, that generalizes simple g-mechanisms.

I. Introduction

We consider the problem of allocating a fixed amount of an infinitely divisible good among self-interested and competing users. In recent years market mechanisms for divisible goods have been extensively studied from a game-theoretic view point (see [9] for a survey on the subject). We refer to the class of noncooperative games derived from such mechanisms as *resource allocation games*; *Nash equilibrium* is usually assumed as a solution concept, and different benchmarks of such markets are studied at their Nash equilibrium point (e.g., see [14], [6], [15], [10], [16], [7], [11], [3], [23], [4]).

In this paper, we focus our attention on markets where users are restricted to communicate their demands via a one-dimensional strategy space. We investigate the fundamental question of why might we expect such market to be in a Nash equilibrium. One explanation of equilibrium is that users' knowledge about the system and about each other is complete, and common knowledge; and their resulting analysis leads them to play according to Nash equilibrium. However, this assumption is unrealistic in large scale markets, e.g., markets for resources on the Internet. In this paper we take the path of explaining Nash equilibrium as the result of repeated interaction by means of market feedback. This is in particular compelling in communication networks, where users interact repeatedly and frequently with network resources.

We search for a *distributed* and *dynamic* process that converges to Nash equilibrium in markets for a single divisible good. A minimal requirement from a distributed process is that users' update rules are *uncoupled*, i.e., each user decides on his actions based on feedback from the market (e.g., price) and his own payoff function; a user is assumed to have no other information regarding the other users in the system, and in particular his actions are oblivious to other users' payoff functions. It is known that no uncoupled dynamic that leads to Nash equilibrium exists in general games [8]. However, such dynamics exist for several game classes, e.g., *zero-sum games* [2], *supermodular games* [17], *concave games* with a strict diagonal concavity property [20], and *potential games* (also known as congestion games) [19].

Resource allocation games do not belong to any of the above game classes, and in particular do not belong to the class of congestion games in general [10]. Congestion games admit a *potential function* [19], which serves as a powerful tool in showing convergence results. The main property of a potential function is that any unilateral deviation by any player yields a payoff change that is identical to the change in the potential. Resource allocation games do not generally admit a potential in the sense of Monderer and Shapley [19], i.e., over the joint strategy space (the space of bid vectors). However, for many interesting examples, it is possible to construct a potential over the *allocation* space.

Our main observation is that the existence of a potential in the allocation space, along with an appropriate feedback from the mechanism, suffices for the existence of an uncoupled dynamic that leads to Nash equilibrium. We develop our result in the context of *proportional allocation* markets [21], where the resource is allocated to users in proportion to their bids.

We consider the continuous-time better response dynamic, where users update their bids in a direction that maximizes their payoff. We show that under appropriate regularity conditions, better response dynamics converge to Nash equilibrium in proportional allocation markets that admit a potential in the allocation space.

We apply our results to the following classes of market mechanisms.

1) *Smooth market-clearing mechanisms*. These are non-discriminatory mechanisms studied in [11], in which a single price is used to clear the market. This class includes the *Kelly mechanism* as a special case [13], for which (to the best of our knowledge) no convergence result was previously known for price anticipating

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users and general concave utilities.[1].

2) *Simple g-mechanisms*. These are market mechanisms that implement an efficient Nash equilibrium, by employing price discrimination [16], [7]. Our results also extend to a subset of the scalar strategy Vickrey-Clarke-Groves (SSVCG) mechanisms, studied in [11], [25]; this subset generalizes simple g-mechanisms.

All proofs are omitted due to space constraints.

II. Related Work

Slade [22] studies potential functions in the strategy space for oligoplies. She shows how a dynamic process that converges to Nash equilibrium can be derived from a potential; she further gives necessary and sufficient conditions for the existence of a potential. Monderer and Shapley [19] extend these ideas to the class of congestion games; they show an equivalence between congestion games and games that admit a potential (potential games).

The Kelly mechanism is a proportional allocation mechanism studied in [13], [12]. In [13] it is shown that when resources are allocated in proportion with payments (the Kelly mechanism), the unique competitive equilibrium maximizes the system welfare (a version of the first fundamental theorem of welfare economics). Kelly, Maulloo, and Tan [12] consider a better response process, that is run simultaneously by the users and by the network (the price-setting entity), and show it converges to the competitive equilibrium point.

Hajek and Gopalakrishnan [6] and Johari and Tsitsiklis [10] observe that the Nash equilibrium of the Kelly mechanism solves a social welfare problem with modified utilities. This suggests a potential function in the allocation space. We note that for the Kelly mechanism, Maheswaran and Basar [15] show that Nash equilibrium is locally stable for a particular dynamic; they remark that simulations suggest the global stability of their process. Our work establishes convergence of a related dynamic analytically, using the modified utility characterization of [10].

We conclude by noting that Even-Dar et al. [5] study a subclass of concave games called *socially concave games*. They show that for socially concave games, if each user follows a learning rule with the *no-external regret* property (cf. [1]), then the average joint action converges to Nash equilibrium. They show that their result applies to the Kelly mechanism for the special case when users' utility for rate is linear.

III. Background

Suppose N users $\{1,\ldots,N\}$ share a communication link of unit capacity. We consider a general setting where users communicate their demand function to a resource manager, and the resource manager allocates shares of the channel capacity to the users, and charges them accordingly. We consider mechanisms where users are restricted to choose from parameterized demand functions, where the parameter is a real scalar. Let θ_i denote the parameter or *bid* of user i, and let $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_N)$ denote the users' bid vector. We denote by $\boldsymbol{\theta}_{-i}$ the vector of all bids other than that of user i.

A *market mechanism* is a pair of mappings $(\boldsymbol{x},\boldsymbol{p})$: an *allocation rule* $\boldsymbol{x}:\mathbb{R}_+^N \to \mathbb{R}_+^N$ that maps a bid vector to the set of feasible allocations $\Delta^N = \{\boldsymbol{y} | \sum_i \hat{y}_i \le 1, \text{and } y_i \ge 0 \text{ for every } i\}$; and a *payment rule* $\boldsymbol{p}:\mathbb{R}_+^N \to \mathbb{R}_+^N$ that maps a bid vector to users' payments. Both $\boldsymbol{x}$ and $\boldsymbol{p}$ are assumed to be differentiable. Each user i receives his share $x_i(\boldsymbol{\theta})$ and is charged $p_i(\boldsymbol{\theta})$. We assume that participation is voluntary; if a user i's bid is $\theta_i = 0$, then his payment is $p_i(0,\boldsymbol{\theta}_{-i}) = 0$, no matter what the others' bid.

One prominent example of a market mechanism is the *Kelly mechanism*, studied in the context of communication networks by Kelly [13]. The resource allocation rule of the Kelly mechanism is the *proportional allocation* rule:

$$x_i(\boldsymbol{\theta}) = \begin{cases} \frac{\theta_i}{\sum_{i=1}^N \theta_i} & \text{if } \theta_i > 0; \\ 0 & \text{if } \theta_i = 0. \end{cases} \tag{1}$$

Each user i pays his bid, i.e., the payment rule is the identity function $p_i(\boldsymbol{\theta}) = \theta_i$. Observe that in this case the price is set equal to the sum of bids. In this work we focus on mechanisms with the proportional allocation rule, but we consider a wide variety of payment functions in addition to that of the Kelly mechanism.

Each user i has a utility for rate x_i equal to $U_i(x_i)$. We assume that for all i, over the domain $x_i \ge 0$ the utility function $U_i(x_i)$ is concave, strictly increasing, continuous, and continuously differentiable (where we interpret $U_i'(0)$ as the right directional derivative at zero). Furthermore, we assume $U_i'(0)$ is finite. Denote by $\mathcal{U}$ the set of all feasible utility functions.

$$\begin{aligned} \mathcal{U} = \big\{ U : \mathbb{R}^+ \to \mathbb{R}^+ | U &\text{ is continuous, strictly,} \\ &\text{increasing, concave on } [0,\infty), \\ &\text{and continuously differentiable on } [0,\infty) \\ &\text{with } U'(0) < \infty \big\} \end{aligned} \tag{2}$$

We denote by $\boldsymbol{U} = (U_1,\ldots,U_N)$ a vector of utility functions, and refer to a pair $(N,\boldsymbol{U})$, as a *utility system*.

In addition we assume that utility is measured in monetary units; thus if user i receives rate x_i and is charged a price μ_i per unit, then his payoff is

$$U_i(x_i) - \mu_i x_i.$$

Users are called *price anticipating* if they anticipate the effect of their actions on the market-clearing price. In this case, each participant views the market-clearing price as a function of the composite strategy vector of all market participants. Thus the competition between market participants who are price anticipating is a game: the payoff of a given player is directly expressed as a function of his own strategy, as well as the strategies of all other players. A user's strategy space

[1] Even-Dar et al. [5] show a convergence result for the special case of linear utilities, and Waslander et al. [24] show such a result for the case of two users with linear utilities.

is the nonnegative real line. Let Q_i denote user i's payoff function:

$$Q_i(\theta) = \begin{cases} U_i(x_i(\boldsymbol{\theta})) - p_i(x_i(\boldsymbol{\theta})) & \text{if } \theta_i > 0; \\ U_i(0) & \text{if } \theta_i = 0. \end{cases} \quad (3)$$

We refer to the resulting game as a *resource allocation game*. A *Nash equilibrium* of a resource allocation game is a vector $\boldsymbol{\theta} \geq \mathbf{0}$ such that

$$Q_i(\theta_i, \boldsymbol{\theta}_{-i}) \geq Q_i(\bar{\theta}_i, \theta_{-i}), \quad \text{for all } \bar{\theta}_i \geq 0.$$

We restrict attention to resource allocation games such that there exists a $B_i > 0$ for each bidder (possibly game-dependent), such that it is never best response to bid more than B_i. Hence, effectively, the strategy space of user i becomes the interval $[0, B_i]$. We make the additional assumption that $Q_i(\theta_i, \boldsymbol{\theta}_{-i})$ is concave in θ_i over the domain $[0, B_i]$, for every fixed θ_{-i}. Applying Rosen's existence theorem we conclude that under these two assumptions a Nash equilibrium exists.

For simplicity of presentation, for the remainder of the paper, *we restrict attention to resource allocation games where there exists a Nash equilibrium which lies in the interior of the joint strategy space.*

IV. Potentials and Better Response Dynamics

In this section we design a distributed algorithm that converges to Nash equilibrium for a large class of market mechanisms. The update rule is a version of *better response dynamics*, where each user updates his bid in the direction that improves his immediate payoff, assuming that the bids of the other users are fixed.

A. Allocation Space Potentials

Our main contribution in this paper is the observation that the existence of a potential function in the allocation space is useful in the design of a distributed, uncoupled dynamic process in the bid space that converges to Nash equilibrium. By a "potential" we mean a separable objective over the users, whose maximizer is a Nash equilibrium allocation of the resource allocation game.

Definition 1 A mechanism $(\boldsymbol{x}, \boldsymbol{p})$ admits an uncoupled potential in the allocation space, if there exists a mapping $\varphi : \mathcal{U} \to \mathcal{U}$, such that $\varphi(U)$ is strictly concave for every $U \in \mathcal{U}$, and for every utility system $(N, \boldsymbol{U})$, any Nash equilibrium of the game $(Q_1, \ldots, Q_N)$ yields an allocation that is the unique maximizer of the following problem:

$$\begin{array}{ll} \text{maximize} & P(\boldsymbol{y}; \varphi) = \sum_i \widehat{U}_i(y_i) \\ \text{subject to} & \boldsymbol{y} \in \mathbb{R}^N, \sum_{i=1}^N y_i \leq 1, \\ & y_i \geq 0 \text{ for all } i, \end{array} \quad (4)$$

where $\widehat{U}_i = \varphi(U_i)$ for each i.

We say that a point $\boldsymbol{y} \in \Delta^N$ is a *potential maximizer* if it solves (4), and by an abuse of notation we refer to a bid vector $\boldsymbol{\theta}$ as a potential maximizer if the allocation $\boldsymbol{x}(\boldsymbol{\theta})$ is a potential maximizer. We refer to the utility $\widehat{U}_i = \phi(U_i)$ as the *modified utility* of user i. Notice that $P(\boldsymbol{y}; \varphi)$ is strictly concave as it is a sum of strictly concave functions.

Remark 2 The modified utility system $(N, \widehat{\boldsymbol{U}})$ is *uncoupled*, in the sense that $\widehat{U}_i$ is a function of U_i, and does not depend on the utility functions of users other than i. In other words, for a fixed market mechanism, the modified utility $\widehat{U}_i$ derived from utility function U_i is the same for every $N \geq 1$, and every $\boldsymbol{U}_{-i} \in \mathcal{U}^{N-1}$. We employ this uncoupledness in the design of a distributed dynamics that converges to Nash equilibrium.

Transforming an equilibrium problem to an optimization problem has proven fruitful, as methods for solving optimization problems are more advanced than methods for solving equilibrium problems. Slade [22] shows necessary and sufficient conditions for the existence of a potential in oligopolies; given an N-player oligopoly game with payoff functions $(R_1, \ldots, R_N)$ and where s_i denotes user i's strategy, Slade searches for a potential $F(s_1, \ldots, s_N)$ such that:

$$\frac{\partial}{\partial s_i} F(\boldsymbol{s}) = \frac{\partial}{\partial s_i} R_i(\boldsymbol{s}), \text{ for every } i. \quad (5)$$

Notice the similarity between (5) and Definition 1; in particular, observe that we have:

$$\frac{\partial}{\partial y_i} P(\boldsymbol{y}; \varphi) = \frac{d}{dy_i} \widehat{U}_i(y_i) \text{ for every } i.$$

Monderer and Shapley [19] show that every congestion game admits a potential. It is well known that the existence of a potential suffices for the convergence to Nash equilibrium of different dynamic processes, e.g., best response [19], better response [22], and perturbed fictitious play [18]. However, it can be verified that no potential exists for general resource allocation games, even in the simple case of the Kelly mechanism [13].

The difficulty with the analysis of these dynamic rules for resource allocation games is that the best response functions of resource allocation games are usually ill behaved in the bid space. On the other hand in many examples, these functions are well behaved in the allocation space, and admit a potential in that space (cf. Definition 1). The existence of a potential in the allocation space gives rise to numerical algorithms for computing a Nash equilibrium allocation, e.g., gradient ascent and Newton method. However, a resource allocation game is played over the bid space, and so a process in the allocation space is in itself insufficient for the construction of a distributed algorithm that converges to a Nash equilibrium.

We next define a *virtual price function* for the vector of modified utilities of Definition 1; our main result shows that the existence of a virtual price function gives rise to an uncoupled dynamics that converges to Nash equilibrium in the bid space, as well as the allocation space.

Definition 3 Given a mechanism $(\boldsymbol{x}, \boldsymbol{p})$ that admits an uncoupled potential $P(\cdot; \varphi)$ in the allocation space, a real valued function $\mu(\boldsymbol{\theta}) : \mathbb{R}_+^N \to \mathbb{R}_+$ is a *virtual price function* for $(\boldsymbol{x}, \boldsymbol{p})$ if for every utility system $(N, \boldsymbol{U})$:

$$\begin{aligned} \widehat{U}_i'(x_i(\theta_i^{\text{NE}}, \boldsymbol{\theta}_{-i}^{\text{NE}})) &= \mu(\boldsymbol{\theta}^{\text{NE}}) \quad \text{if } \theta_i > 0; \\ \widehat{U}_i'(x_i(\theta_i^{\text{NE}}, \boldsymbol{\theta}_{-i}^{\text{NE}})) &\leq \mu(\boldsymbol{\theta}^{\text{NE}}) \quad \text{if } \theta_i = 0, \end{aligned} \tag{6}$$

where $\widehat{U}_i$ is user $i's$ modified utility as in Definition 1, and $\boldsymbol{\theta}^{\text{NE}}$ is a potential maximizer.

In this paper we consider mechanisms with virtual price functions that satisfy some additional regularity conditions. In particular, we make the assumption that the virtual price is a function of the aggregate bid, denoted by:

$$\Theta = \sum_{i=1}^{N} \theta_i.$$

By an abuse of notation we write $\mu(\Theta)$ for the virtual price function. In addition we require that $\mu(\Theta)$ is continuous and strictly increasing in the aggregate bid, with $\mu(0) = 0$.

In the sequel we require the following lemma.

Lemma 4 *Suppose we are given a market $(\boldsymbol{x}, \boldsymbol{p})$ with a proportional allocation rule that admits an uncoupled potential $P(\cdot; \varphi)$, with virtual price function $\mu(\Theta)$ that is continuous and strictly increasing in the aggregate bid, with $\mu(0) = 0$.*

Then there exists a unique potential maximizer Nash equilibrium $\boldsymbol{\theta}^{\text{NE}}$.

In the next sections we show that for markets with proportional allocation rules, the existence of such a virtual price function suffices for the existence of an update rule that converges to a Nash equilibrium in the bid space.

B. Better Response Dynamic

We now describe the dynamic process we consider. An update rule considered most often in the literature on dynamics and learning in games is the *best response* dynamics, where at each step a user optimizes his action with respect to the actions of the other users in the previous step, assuming they will not change. However, it has been previously observed that the best response dynamics need not converge to a Nash equilibrium in resource allocation games [5], [15], [24].

A related update rule is the *better response dynamics*, where users update their bid in the direction of the best response bid, but with a small step size. In this paper we assume time is continuous, and analyze a particular better response dynamics.

We define a better response dynamics for each user with respect to his modified utility and the virtual price function. Let $\boldsymbol{\theta}(t)$ denote the vector of bids at time t, and let $\boldsymbol{y}(t) \in \Delta^N$ denote the allocation vector at time t. The better response dynamics we consider updates the bid in the direction of the difference between the marginal virtual utility and the virtual price, and in proportion to the allocation:

$$\dot{\theta}_i(t) = y_i(t) \left(\widehat{U}_i'(y_i(t)) - \mu(\boldsymbol{\theta}(t)) \right). \tag{7}$$

C. Convergence to Nash Equilibrium

By definition, a potential maximizer Nash equilibrium is a rest point for (7). We now show that the existence of an uncoupled potential and a corresponding virtual price function $\mu(\Theta)$ that depends on the sum of the bids Θ is sufficient for the convergence of (7), for the special case of the proportional allocation rule in (1). In this case (7) converges to a Nash equilibrium which is a potential maximizer.

Theorem 5 *Suppose we are given a market $(\boldsymbol{x}, \boldsymbol{p})$ with a proportional allocation rule that admits a potential $P(\cdot; \varphi)$ in the allocation space, and a corresponding virtual price function $\mu(\Theta)$, such that $\mu(\Theta)$ is a continuous and strictly increasing function over $[0, \infty)$ with $\mu(0) = 0$.*

Then the better response dynamics in (7) converges to the unique potential maximizer Nash equilibrium $\boldsymbol{\theta}^{\text{NE}}$ (cf. Lemma 4) as $t \to \infty$, from any initial point $\boldsymbol{\theta}(0) > \mathbf{0}$.

The proof of Theorem 5 relies on the analysis of the following dynamic process in the allocation space, where each user updates his allocation in the direction of the difference between his marginal modified utility, and the average marginal modified utility. Formally, for each i, suppose:

$$\dot{y}_i(t) = \zeta(t) y_i(t) \left(\widehat{U}_i'(y_i(t)) - \sum_j y_j(t) \widehat{U}_j'(y_j(t)) \right), \tag{8}$$

where $\zeta(t)$ is any integrable, real-valued, strictly positive function. (If $\Theta(t) > 0$ for all t, we can derive (8) with $\zeta(t) = 1/\Theta(t)$ from (7) by the relation $y_i(t) = \theta_i(t)/\Theta(t)$.) Let $I(\Delta^N)$ denote the interior of feasible allocations $I(\Delta^N) = \{\boldsymbol{y} \mid \sum_i y_i = 1, y_i > 0 \text{ for all } i\}$. We next show that the potential maximizer allocation is a globally stable point for (8) over the interior of the feasible allocation set.

Lemma 6 *The potential maximizer Nash equilibrium $\boldsymbol{y}^{\text{NE}}$ is globally stable for the dynamical system (8), for any initial point in the domain $I(\Delta^N)$.*

A major step in the proof of Theorem 5 is to show that the dynamic of the aggregate bid $\Theta(t)$

$$\dot{\Theta}(t) = \sum_i y_i(t) \widehat{U}_i'(y_i(t)) - \mu(\Theta(t)),$$

leads to the Nash equilibrium aggregate bid $\Theta^{\text{NE}} = \sum_i \theta_i^{\text{NE}}$. In combination with Lemma 6, this leads to the convergence of $\boldsymbol{\theta}(t)$ to the unique potential maximizer Nash equilibrium $\boldsymbol{\theta}^{\text{NE}}$.

V. Applications

In this section we give three examples of classes of market mechanisms that admit a virtual price function. The first is the class of smooth market-clearing mechanisms, that covers essentially all market-clearing mechanisms [11] with no price discrimination, under some regularity conditions. The second example is the class of simple g-mechanisms, studied in [16], [7]. The third example is a subset of the class of SSVCG mechanisms [11], also known as VCG-Kelly mechanisms; the subset we consider generalizes the class of simple g-mechanisms.

A. Smooth Market-Clearing Mechanisms

A smooth market-clearing mechanism [11] allocates the entire unit capacity using a single non-discrimnatory price.

Definition 7 (Johari and Tsitsiklis [11]) A smooth market-clearing mechanism for one unit of infinitely divisible good is a differentiable function $D : (0, \infty) \times [0, \infty) \to \mathbb{R}_+$ such that for all $\boldsymbol{\theta} \in \mathbb{R}_+^N$, there exists a unique solution $p > 0$ to the following equation:

$$\sum_{i=1}^{N} D(p, \theta_i) = 1. \tag{9}$$

We let $p_D(\boldsymbol{\theta})$ denote this solution.

Given $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)$, the payoff of a user i is

$$Q_i(\boldsymbol{\theta}) = U_i(D(p_D(\boldsymbol{\theta}), \theta_i)) - D(p_D(\boldsymbol{\theta}), \theta_i)p_D(\boldsymbol{\theta}), \tag{10}$$

if the user is price anticipating.

For technical reasons Johari and Tsitsiklis restrict attention to the following subset $\mathcal{D}$ of smooth market-clearing mechanisms. We make the same restriction.

Definition 8 (Johari and Tsitsiklis [11]) The class $\mathcal{D}$ consists of all smooth market-clearing mechanisms D such that the following conditions are satisfied:

1) For all $U \in \mathcal{U}$, a user's payoff is concave if he is price anticipating; that is, for all N, and for all $\boldsymbol{\theta}_{-i} \in (\mathbb{R}_+)^N$, the function: $Q_i(\theta_i, \boldsymbol{\theta}_{-i})$ is concave in $\theta_i > 0$ if $\boldsymbol{\theta}_{-i} \neq 0$, and concave in $\theta_i \geq 0$ if $\boldsymbol{\theta}_{-i} = 0$.
2) The demand functions are nonnegative; i.e., for all $p > 0$ and $\theta \geq 0$, $D(p, \theta) \geq 0$.

To show that a smooth market-clearing mechanism admits an uncoupled potential and a corresponding virtual price function, we employ the notion of a *price function* defined by Maheswaran and Basar [15] in the context of a resource allocation game. For a user i, a price function maps an allocation y_i to the price that i pays when he is allocated y_i, and his bid θ_i is a best response to $\boldsymbol{\theta}_{-i}$. A price function does not necessarily exist for every market mechanism. Maheswaran and Basar [15] show that a price function exists for the Kelly mechanism. Lemma 9 extends this result to smooth market-clearing mechanisms.

Lemma 9 *For every smooth market-clearing mechanism $D \in \mathcal{D}$, and user utility $U_i \in \mathcal{U}$, and for all y_i, there exists a unique value $\gamma_i(y_i)$ such that if θ_i is a best response to $\boldsymbol{\theta}_{-i}$ and $y_i = D(p_D(\boldsymbol{\theta}), \theta_i)$, then $\gamma_i(y_i) = p_D(\boldsymbol{\theta})$. Further, $\gamma_i(y_i)$ is strictly decreasing in y_i.*

Given a utility function $U \in \mathcal{U}$, let $\gamma(\cdot; U)$ be the corresponding function given by Lemma 9. Given a smooth market-clearing mechanism $D \in \mathcal{D}$, we define a map $\varphi : \mathcal{U} \to \mathcal{U}$ by:

$$\varphi(U)(y) = \int_0^y \gamma(z; U)dz.$$

Using Lemma 9 it is straightforward to show that $P(\cdot; \varphi)$ is an uncoupled potential (cf. Definition 1), and that $\mu(\Theta) = \Phi(\Theta)$ serves as a corresponding virtual price function.

Theorem 10 *For every smooth market-clearing mechanism $D \in \mathcal{D}$, the function $P(\cdot; \varphi)$ is an uncoupled potential, with a virtual price function $\mu(\Theta) = \Phi(\Theta)$.*

As an example of the preceding theorem, consider the special case of the Kelly mechanism. This mechanism can be shown to be equivalent to a smooth market-clearing mechanism where $D(p, \theta) = \theta/p$. Further, for this mechanism, $\Phi(\Theta) = \Theta$, and $\varphi(U)$ is given by:

$$\varphi(U)(y) = (1 - y)U(y) + \int_0^y U(z)dz.$$

(See [11] for these results.) The preceding expression is exactly the modified utility function of [6], [10]. Thus our result in Theorem 5 provides an uncoupled dynamic that converges to the unique Nash equilibrium for the Kelly mechanism with general utilities.

B. Simple *g*-Mechanisms

A second application is to the class of simple g-mechanisms [16], [7]. Given a continuous and strictly increasing surjection $g : \mathbb{R}_+ \to \mathbb{R}_+$, a *simple g-mechanism* is a market with a proportional allocation rule, and with payment rule:

$$p_i(\boldsymbol{\theta}) = \begin{cases} \sum_{j\neq i} \theta_j \int_{z=0}^{\theta_i} \frac{g(t+\sum_{j\neq i}\theta_j)}{\left(t+\sum_{j\neq i}\theta_j\right)^2} dt & \text{if } \theta_i > 0; \\ 0 & \text{if } \theta_i = 0. \end{cases} \tag{11}$$

In [7], Hajek and Yang establish that a simple g-mechanism admits an uncoupled potential by showing that the sum of utilities is maximized at a Nash equilibrium allocation; in addition it follows from the Nash equilibrum first order conditions that $g(\Theta)$ is a virtual price function.

Lemma 11 *A simple g-mechanism admits the total utility, i.e., $P(\cdot; \varphi)$ with $\varphi(U) = U$, as an uncoupled potential, with $g(\Theta)$ as a virtual price function.*

Hajek and Yang [7] have addressed the global stability of Nash equilibria of simple g-mechanisms. They consider the following differential inclusion:

$$\dot{\theta} \in \theta_i(t) \cdot \overline{\text{sgn}}\left(U_i'(y_i(t)) - g(\Theta(t))\right), \tag{12}$$

where $\overline{\text{sgn}}$ is defined as follows:

$$\overline{\text{sgn}}(y) \in \begin{cases} \{-1\}, & \text{if } y < 0 \\ [-1, 1], & \text{if } y = 0 \\ \{1\}, & \text{if } y > 0 \end{cases} \tag{13}$$

Hajek and Yang show that (12) converges to the unique efficient Nash equilbirium, from any initial point on the interior of the joint bid space. Observe that the dynamical system in (12) is different than the one in (7) since the rate at which the bid is updated in (7) is sensitive to the distance between the marginal modified utility and the virtual price.

C. Scalar Strategy VCG Mechanisms

Scalar strategy VCG (SSVCG) mechanisms are an adaptation of the VCG mechanism to the case of one-dimensional strategy spaces [11]; they are also known as *VCG-Kelly* mechanisms [25]. Users are allowed to choose from a given single parameter family of utility functions $\overline{U}(\cdot,\theta)$, parameterized by $\theta \in (0,\infty)$. It is assumed that the function $\overline{U}(\cdot,\theta) : y \mapsto \overline{U}(y,\theta)$, defined for $y \geq 0$, is strictly concave, strictly increasing, and continuously differentiable for $y \geq 0$.

Given $\boldsymbol{\theta}$, the mechanism chooses $\boldsymbol{x}(\boldsymbol{\theta})$ such that:

$$\begin{array}{ll} \boldsymbol{x}(\boldsymbol{\theta}) = \mathrm{argmax} & \sum_i \overline{U}(y_i,\theta_i) \\ \text{subject to} & \boldsymbol{y} \in \mathbb{R}^N, \sum_{i=1}^N y_i \leq 1, \\ & y_i \geq 0 \text{ for all } i. \end{array} \quad (14)$$

Since $\overline{U}(\cdot,\theta_i)$ is strictly concave for each i, $\boldsymbol{x}(\boldsymbol{\theta})$ is uniquely determined.

The monetary payment by user i is:

$$p_i(\theta) = -\sum_{j\neq i} \overline{U}(x_j(\boldsymbol{\theta}),\theta_j) + h_i(\boldsymbol{\theta}_{-i}) \quad (15)$$

for some function h_i.

Let $\lambda(\boldsymbol{\theta})$ denote the Lagrange multiplier of the capacity constraint in (14). We consider the special case of SSVCG mechanisms where $\lambda(\boldsymbol{\theta}) = \Psi(\Theta)$, where $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and differentiable function of the aggregate bid Θ. For SSVCG mechanisms Johari and Tsitsiklis show the existence of an efficient Nash equilibrium; in fact, Johari and Tsitsiklis show that the class of simple g-mechanisms is a proper subset of the class of SSVCG mechanisms [11].

For the next result we consider a subset of SSVCG mechanisms that go beyond proportional allocation. We consider *semi-proportional* allocation rules, of the form:

$$x_i(\boldsymbol{\theta}) = \begin{cases} f\left(\frac{\theta_i}{\sum_{i=1}^N \theta_i}\right) & \text{if } \theta_i > 0; \\ 0 & \text{if } \theta_i = 0 \end{cases} \quad (16)$$

for some strictly increasing, continuous surjection $f : \mathbb{R} \to \mathbb{R}$; i.e., the share of each user is a function of his proportional share.

If the efficient Nash equilibrium is unique, then the dynamical system (7) converges to the unique Nash equilibrium.

Proposition 12 For any SSVCG mechanism such that: (1) the Lagrange multiplier of the problem 14 depends only on $\Theta = \sum_i \theta_i$, i.e., $\lambda(\boldsymbol{\theta}) = \Psi(\Theta)$; (2) the allocation rule is semi-proportional; and (3) the Nash equilibrium is unique, the total utility (i.e., $P(\cdot;\varphi)$ with $\varphi(U) = U$) is an uncoupled potential, with virtual price function $\Psi(\Theta)$.

We note that some alterations to the proof of Theorem 5 are needed to handle semi-proportional allocation rules..

Corollary 13 *Under the conditions of Proposition (12), the better response dynamics (7) with $\mu(\Theta) = \Psi(\Theta)$ converges to the unique Nash equilibrium.*

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