

# Modularity and Greed in Double Auctions<sup>☆</sup>

Paul Dütting<sup>a</sup>, Inbal Talgam-Cohen<sup>b</sup>, Tim Roughgarden<sup>b</sup>

<sup>a</sup>*Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, UK.*

<sup>b</sup>*Department of Computer Science, Stanford University, 353 Serra Mall, Stanford, CA 94305, USA.*

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## Abstract

Designing double auctions is a complex problem, especially when there are restrictions on the sets of buyers and sellers that may trade with one another. The goal of this paper is to develop “black-box reductions” from double-auction design to the exhaustively-studied problem of designing single-sided mechanisms.

We consider several desirable properties of a double auction: feasibility, dominant-strategy incentive-compatibility, the still stronger incentive constraints offered by a deferred-acceptance implementation, exact and approximate welfare maximization, and budget-balance. For each of these properties, we identify sufficient conditions on the two one-sided mechanisms—one for the buyers, one for the sellers—and on the method of composition, that guarantee the desired property of the double auction.

Our framework also offers new insights into classic double-auction designs, such as the VCG and McAfee auctions with unit-demand buyers and unit-supply sellers.

*Keywords:* Mechanism Design, Double Auctions, Trade Reduction Mechanism, Deferred-Acceptance Auctions

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## 1. Introduction

Double auctions play an important role in mechanism design theory and practice. They are of theoretical importance because they solve the fundamental problem of how to organize trade between a set of buyers and a set of sellers, when both the buyers and the sellers act strategically. Important practical applications include the New York Stock Exchange (NYSE), where buyers and sellers trade shares, and the upcoming spectrum auction conducted by the US Federal Communication Commission (FCC), which aims at reallocating spectrum licenses from TV broadcasters to mobile communication providers [29].

Designing double auctions can be a complex task, with several competing objectives. These include, but are not limited to: feasibility, dominant-strategy incentive compatibility (DSIC), the still stronger incentive constraints offered by a deferred-acceptance implementation such as weak group-strategyproofness (WGSP) or implementability as a clock auction [28], exact or approximate welfare maximization, and budget balance (BB).

Perhaps the cleanest approach to designing double auctions for complex settings is via a “black-box reduction” to the exhaustively-studied problem of designing single-sided mechanisms. The goal of this paper is to develop the theory that explains when and how such black-box reductions work.

### 1.1. Motivating Examples

Suppose there are  $n$  buyers and  $m$  sellers. Each buyer  $i$  wants to acquire one unit of an identical good, and has a value  $v_i$  for it. Each seller  $j$  produces one unit of the good, and producing it incurs a cost of  $c_j$ .

Assume first there are no restrictions on which buyers and sellers can trade with one another. Is it possible, by composing two single-sided mechanisms, to implement the Vickrey-Clarke-Groves (VCG) mechanism [40, 11, 19]

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*Email addresses:* p.d.duetting@lse.ac.uk (Paul Dütting), italgam@cs.stanford.edu (Inbal Talgam-Cohen), tim@cs.stanford.edu (Tim Roughgarden)

that maximizes welfare and is DSIC? What about McAfee’s trade reduction mechanism [27], which accepts all buyer-seller pairs from the welfare-maximizing solution except for the least valuable one, and is DSIC and BB?<sup>1</sup>

The answer to these questions is “yes”. We can implement the VCG mechanism using simple greedy algorithms that sort the buyers by non-increasing value and the sellers by non-decreasing cost. We iteratively query these algorithms for the next buyer-seller pair and accept it if the buyer has a larger value  $v_i$  than the seller’s cost  $c_j$ , and we apply threshold payments. For McAfee’s trade reduction mechanism we use reverse versions of these algorithms, that sort the players by non-decreasing value and non-increasing cost. We iteratively query these algorithms for the next buyer-seller pair and reject it if none of the previously inspected buyer-seller pairs had non-negative gain from trade  $v_i - c_j \geq 0$ , and we again apply threshold payments.

Now, *what happens if we add feasibility constraints on which buyers and sellers can trade?* Such feasibility constraints can be important in practical applications, and their potential richness mirrors the richness of real-life economic settings; see, for example, the recent work on the proposed FCC double auctions for spectrum [29, 32, 25, 26].

As a first example, consider the variation of the above problem in which the buyers belong to one of three categories (e.g., they are firms that are either small, medium, or large in size). To ensure diversity among buyers, the policy maker requires that no more than  $k_i$  buyers from each category  $i$  shall be accepted (for additional quota examples see, e.g., [20]).

In this example it is still possible to implement the VCG and trade reduction mechanisms by composing two one-sided mechanisms. The only change is to the mechanism used for the buyers. In its forward version we would go through the buyers in order of their value and accept the next buyer if and only if we haven’t already accepted  $k_\ell$  buyers from that buyer’s category  $\ell$ . In its backward version we would go through the buyers in reverse order and reject the next buyer unless there are  $k_\ell$  or fewer buyers from that category left.

As a second example, consider the variant of the original (unconstrained) problem in which sellers have one of two “sizes,”  $s$  or  $S$ , where  $S > s$ . For example, agents could be firms that have either a small or a large facility, or that pollute the environment to different extents. Suppose there is a cap  $C$  on the combined size of the sellers that can be accepted. For additional packing examples see, e.g., [1].

In this example it is less clear what to do. Even putting aside our goal of a reduction to one-sided mechanism design, computing the welfare-maximizing solution is an NP-hard packing problem [e.g., 23], and so specifically the one-sided greedy-by-cost mechanism is no longer optimal. We thus shift our attention to approximately-maximizing solutions, but it is not clear which one-sided approximation methods — greedy according to cost, greedy according to cost divided by size, non-greedy algorithms, etc. — would offer good approximation guarantees in the double auction context, where the choices of buyers and sellers are entangled. Furthermore, it is not clear if the good properties of the double-sided VCG and McAfee mechanisms, such as DSIC or BB, would continue to hold.

## 1.2. Approach and Results

We advocate a modular approach to the design of double auctions, which is applicable to complex feasibility constraints on both sides of the market.

This approach breaks the design task into two subtasks: (a) the design of two one-sided mechanisms and (b) the design of a composition rule that pairs buyers and sellers. To identify what we want from the respective subtasks, we prove a number of *compositions theorems* of the general form:

If the one-sided mechanisms  $\mathcal{M}_1$  and  $\mathcal{M}_2$  have properties  $A_1$  and  $A_2$  and the composition rule has property  $B$ , then the resulting double auction has property  $C$ .

A main theme of this work is thus to identify sufficient conditions on the two one-sided mechanisms and on the method of composition that guarantee a desired property of the double auction.

We start with sufficient conditions that ensure that the double auction is DSIC, resp., has the stronger incentive properties shared by deferred-acceptance implementations [29] (generalizing the Gale-Shapley mechanism [17]).

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<sup>1</sup>For simplicity we focus on a version of McAfee’s trade reduction mechanism in which the least valuable pair is always rejected. Cf. the full version which is defined as follows: Sort buyers by non-increasing value  $v_1 \geq v_2 \geq \dots$  and sellers by non-decreasing cost  $c_1 \leq c_2 \leq \dots$ . Let  $k$  be the largest index such that  $v_k \geq c_k$ . Compute  $t = (v_{k+1} + c_{k+1})/2$ . If  $t \in [c_k, v_k]$  let buyers/sellers  $1, \dots, k$  trade with each other. Otherwise exclude the buyer-seller pair with the  $k$ -th highest value and the  $k$ -th lowest cost from trade.

Interestingly, monotonicity of all involved components is not sufficient for DSIC. We also need that the one-sided mechanisms return the players in order of their quality, and for this reason greedy approaches play an essential role in our designs. In the above examples, the greedy-by-value and greedy-by-cost algorithms have this property, while a greedy algorithm based on cost divided by size may violate it. An important consequence of the sufficient conditions we obtain is that trade reduction-style mechanisms can be implemented within the deferred-acceptance framework and therefore share the stronger incentive properties of mechanisms within this class.

We then identify conditions that ensure the double auction obtains a certain fraction of the optimal welfare. These conditions ask that the one-sided mechanisms achieve a certain approximation ratio “at all times”—the intuition being that the final number of accepted players is extrinsic (it depends on the interplay with the other side of the market), and so the mechanism should be close to optimal for any possible number of accepted players, rather than just for the final number of accepted players. We analyze such guarantees for a number of algorithms, including the greedy-by-value and greedy-by-cost algorithms used in the examples above.

We complement the above results with a lower bound on the welfare obtainable by any WGSP mechanism (based on composition or not). We conclude that in some cases, including the unconstrained setting and the setting with diversity constraints discussed above, the trade reduction mechanism cannot only be implemented via composition, but is also minimizes the worst-case welfare loss subject to WGSP.

The last property we consider is BB. Here we show that the same conditions on the one-sided mechanisms and the composition that enable implementation within the deferred-acceptance framework also lead to BB. We complement this result with a lower bound on the welfare achievable by any BB double auction. We again conclude that in several settings, the trade reduction mechanism minimizes the worst-case welfare loss subject to BB.

### 1.3. Applications

To demonstrate the usefulness of our modular approach, we use it to design novel double auctions for problems with non-trivial feasibility structure. We focus on three types of feasibility constraints. These serve to illustrate our design framework in several concrete settings, and are not an exhaustive list of the applications of our results. We describe them somewhat informally below, and formally in Section 2.1.

1. Matroids:<sup>2</sup> The set of feasible subsets of players is downward-closed (if a set  $S$  is feasible any subset  $T \subseteq S$  is feasible) and satisfies an exchange axiom (if two sets  $S, T$  are feasible and  $T$  is larger than  $S$ , then there must be an element  $i \in T \setminus S$  such that  $S \cup \{i\}$  is feasible). The unconstrained problem discussed above as well as the problem with diversity constraints are special cases of this category.
2. Knapsacks: Each player has a size and a set of players is feasible if their combined size does not exceed a given threshold. The variation of the unconstrained problem in which sellers have one of two distinct sizes is a special case of this constraint.
3. Matchings: We are given a graph such that each player corresponds to an edge in this graph, and a set of players is feasible if it corresponds to a matching in this graph. A concrete example of this constraint is a setting where the seller side of the market corresponds to pairs of firms producing complementary goods required to provide a certain service [cf., 34].

Intuitively, the first setting is precisely the setting in which greedy by quality is optimal. The second and third settings can be thought of as different relaxations of the matroid constraint, in which greedy by quality is not optimal but often performs well.

Our framework yields novel VCG- and trade reduction-style mechanisms for all three settings that are either DSIC, or WGSP, implementable as a clock auction and BB, respectively. It also translates approximation guarantees for greedy algorithms into welfare guarantees for these double auctions. The guarantees show that the welfare degrades gracefully as we move away from settings in which greedy is optimal.

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<sup>2</sup>Matroid structure corresponds to an economic substitutability condition referred to as *players are substitutes* in the related literature (e.g., in [41]). This condition requires that the welfare—the total value of all buyers or minus the total cost of all sellers—is a submodular function of the set of buyers or sellers, which is the the case for matroid feasibility constraints.

#### 1.4. Further Related Work

The design principle of modularity is embraced in a diverse range of complex design tasks, from mechanical systems through software design to architecture [4]. Splitting a complex design task into parts or modules, addressing each separately and then combining the modules into a system helps make the design and analysis tractable and robust. Economic mechanisms that operate in complex incentive landscapes while balancing multiple objectives are natural candidates for reaping the benefits of modularity. As far as we know, the only predecessor of our work that explicitly applies a modular approach to a mechanism design problem is Mu'alem and Nisan [30]. They consider only one-sided settings, while we focus on two-sided settings.

Most prior work on double auctions is motivated by the impossibility results of [21] and [31], which state that optimal welfare and BB cannot be achieved simultaneously subject to DSIC or even Bayes-Nash incentive compatibility (BIC). One line of work escapes this impossibility by relaxing the efficiency requirement. This direction can be divided into mechanisms that are BIC and mechanisms that are DSIC. An important example of the former is the buyer's bid double auction of Satterthwaite and Williams [37], Rustichini et al. [36], Satterthwaite and Williams [38], which sets a single price to equate supply and demand. More recent work that falls into this category is [12, 16]. A prominent example of the latter is McAfee's trade reduction mechanism, which disallows all but the least efficient trade. This mechanism has been generalized to more complex settings in [2, 8, 18, 3, 10]. More recent work that falls into this category is [24, 6] (where [24] actually applies *ex post* incentive compatibility, as appropriate for interdependent values). A second line of work that seeks to escape the impossibility results was recently initiated by [9], by analyzing the trade-off between incentives and efficiency while insisting on budget balance. Our work is different in that it adds to the double auction design problem the objectives of feasibility and WGSP, and takes an explicitly modular approach to achieve the objectives.

The WGSP property that we highlight was studied in detail in [22], although a complete characterization of WGSP mechanisms is not known. Deferred-acceptance algorithms on which part of our work is based are proposed in [29], and their performance is analyzed in [14]. Our work extends the deferred-acceptance framework from one-sided settings to two-sided settings.

The greedy approach has been extensively studied in the context of one-sided mechanism design, for both single- and multi-parameter settings; see, e.g., [7] and references within.

#### 1.5. Paper Organization

Section 2 covers preliminaries of the settings to which our framework applies, and formally defines properties of double auctions that we are interested in, including incentive compatibility of different types (DSIC and WGSP), welfare-maximization, and BB; this section can be skipped by the expert. Section 3 describes our composition framework: First we define the one-sided mechanisms, and then we turn to different methods of composing these one-sided mechanisms.

The next three sections are roughly organized by the desired double-auction property. Section 4 proves our DSIC and WGSP composition theorems. Section 5 has a similar analysis gives our welfare composition theorem. Section 6 proves our BB composition theorem. Finally, Section 7 studies the interplay of welfare, incentives, and budget-balance.

## 2. Problem Statement

This section defines the double auction settings and the properties of double auction mechanisms that we are interested in. We also single out three settings that will serve as running examples.

### 2.1. Double Auction Settings

We study single-parameter double auction settings. These are two-sided markets, with  $n$  buyers on one side of the market and  $m$  sellers on the other. There is a single kind of item for sale. The buyers each want to acquire a single unit of this item, and the sellers each have a single unit to sell. Each buyer  $i$  has a value  $v_i \geq 0$ , and each seller  $j$  has a cost  $c_j \geq 0$ . We denote by  $\vec{v}$  ( $\vec{c}$ ) the value profile (cost profile) of all buyers (sellers). The players' utilities are quasi-linear, i.e., buyer  $i$ 's utility from acquiring a unit at price  $p_i$  is  $v_i - p_i$ , and seller  $j$ 's utility from selling his unit for payment

$p_j$  is  $p_j - c_j$ . The *welfare* achieved by a set of buyers  $B$  and sellers  $S$  is the difference between the total value of the  $\min\{|B|, |S|\}$ -highest buyers and the total cost of the  $\min\{|B|, |S|\}$ -lowest sellers.

A set of buyers and sellers is *feasible* if the set of buyers is feasible and the set of sellers is feasible and there are at least as many sellers as there are buyers. The sets of buyers that are feasible are expressed as a set system  $(N, \mathcal{I}_N)$ , where  $N$  is the set of all  $n$  buyers, and  $\mathcal{I}_N \subseteq 2^N$  is a non-empty collection of all the feasible buyer subsets. Similarly, feasible seller sets are given as a set system  $(M, \mathcal{I}_M)$ , where  $M$  is the set of all  $m$  sellers, and  $\mathcal{I}_M \subseteq 2^M$  is a non-empty collection of all the feasible seller subsets. The set systems that we consider are *downward closed*, meaning that for every nonempty feasible set, removing any element of the set results in another feasible set. We assume that the two feasibility set systems are publicly known and can be accessed via feasibility oracles. The *welfare subject to feasibility* achieved by a set of buyers  $B$  and sellers  $S$  is the maximum over all welfares achieved by feasible subsets  $B' \subseteq B$  and  $S' \subseteq S$ .

*Running Examples.* We now define formally the examples mentioned and motivated in Section 1.3. We denote by  $U$  the *ground set* of players (either the buyers or sellers in our context), and by  $\mathcal{I}$  the collection of feasible subsets.

1. *Matroids:* A set system  $(U, \mathcal{I})$  is a matroid if (1)  $\emptyset \in \mathcal{I}$ , (2) for all  $S \subset T \subseteq U : T \in \mathcal{I}$  implies  $S \in \mathcal{I}$  (downward closed property), (3) if  $S, T \in \mathcal{I}$  and  $|T| > |S|$ , then there exists  $u \in T \setminus S$  such that  $S \cup \{u\} \in \mathcal{I}$  (exchange property). The sets in  $\mathcal{I}$  are called *independent*. A maximal independent set is called a *basis*, and a minimal dependent set is called a *circuit*.
2. *Knapsacks:* In this case, the elements of the ground set  $U$  have publicly-known *sizes*  $(s_1, \dots, s_{|U|})$ , and the family of feasible sets  $\mathcal{I}$  includes every subset  $S \subseteq U$  such that its total size  $\sum_{i \in S} s_i$  is at most the capacity  $C$  of the knapsack. We denote the ratio between the size of the largest element and the size of the knapsack by  $\lambda \leq 1$ , and the ratio between the size of the smallest element and the size of the largest element by  $\mu \leq 1$ .
3. *Matchings:* A third class of feasibility restrictions are bipartite matching constraints. In this case the ground set  $U$  is the edge set of some bipartite graph  $G = (V, U)$ , and the family of feasible sets  $\mathcal{I}$  are the subsets of the ground set that correspond to bipartite matchings in this graph.

## 2.2. Double Auction Mechanisms

We study direct and deterministic double auction mechanisms, which consist of an *allocation rule*  $x(\cdot, \cdot)$  and a *payment rule*  $p(\cdot, \cdot)$ . The allocation rule takes a pair of value and cost profiles  $\vec{v}, \vec{c}$  as input, and outputs the set of players who are *accepted*, or *allocated*, for trade. For every buyer  $i$  (seller  $j$ ),  $x_i(\vec{v}, \vec{c})$  (resp.,  $x_j(\vec{v}, \vec{c})$ ) indicates whether he is allocated by the mechanism. The payment rule also takes a pair of value and cost profiles  $\vec{v}, \vec{c}$  as input, and computes payments that it charges the buyers and pays to the sellers. We use  $p_i(\vec{v}, \vec{c})$  to denote the payment buyer  $i$  is charged, and  $p_j(\vec{v}, \vec{c})$  to denote the payment seller  $j$  is paid. A buyer who is not accepted is charged 0 and a seller who is not accepted is paid 0. The *welfare* a mechanism achieves is the welfare of its set of accepted players.

*Non-Strategic Properties.* We study the following non-strategic properties of double auction mechanisms:

- *Feasibility.* A double auction mechanism is feasible if for every value and cost profiles  $\vec{v}, \vec{c}$ , the set of accepted buyers and sellers is feasible. Formally, if  $B$  is the set of accepted buyers and  $S$  is the set of accepted sellers, then  $B \in \mathcal{I}_N, S \in \mathcal{I}_M$  and  $|B| \leq |S|$ .
- *Budget balance (BB).* A double auction mechanism is budget balanced if for every value and cost profiles  $\vec{v}, \vec{c}$ , the difference between the sum of payments charged to the accepted buyers and the sum of payments paid to the accepted sellers is non-negative.
- *Efficiency.* For  $\delta \geq 1$ , a double auction mechanism is  $\delta$ -approximately efficient if for every value and cost profiles  $\vec{v}, \vec{c}$ , the welfare it achieves is at least a  $(1/\delta)$ -fraction of the optimal welfare for  $\vec{v}, \vec{c}$  (subject to feasibility).

*Strategic Properties.* We also study the following strategic properties of double auction mechanisms:

- *Individual rationality (IR).* A double auction mechanism is IR if for every value and cost profiles  $\vec{v}, \vec{c}$ , every accepted buyer  $i$  is not charged more than his value  $v_i$ , and every accepted seller  $j$  is paid at least his cost  $c_j$ . Non-accepted players are charged/paid zero.
- *Dominant strategy incentive compatible (DSIC).* A double auction mechanism is DSIC if for every value and cost profiles  $\vec{v}, \vec{c}$  and for every  $i, j, v'_i, c'_j$ , it holds that buyer  $i$  is (weakly) better off reporting his true value  $v_i$  than any other value  $v'_i$ , and seller  $j$  is (weakly) better off reporting his true cost  $c_j$  than any other cost  $c'_j$ . Formally,

$$x_i(\vec{v}, \vec{c})v_i - p_i(\vec{v}, \vec{c}) \geq x_i((v'_i, v_{-i}), \vec{c})v_i - p_i((v'_i, v_{-i}), \vec{c}),$$

and similarly for seller  $j$ .

- *Weak group-strategyproofness (WGSP).* A double auction mechanism is WGSP if for every value and cost profiles  $\vec{v}, \vec{c}$ , for every set of buyers and sellers  $B \cup S$  and every alternative value and cost reports of these players  $v'_B, c'_S$ , there is at least one player in  $B \cup S$  who is (weakly) better off when the players report truthfully as when they report  $v'_B, c'_S$ . Intuitively, such a player does not have a strict incentive to join the deviating group.<sup>3</sup>

The following characterization of DSIC and IR double auction mechanisms follows from standard arguments.

**Proposition 2.1.** *A double auction mechanism is DSIC and IR if and only if:*

1. *The allocation rule is monotone, i.e., for all value and cost profiles  $\vec{v}, \vec{c}$ , every accepted buyer who raises his value remains accepted, and every accepted seller who lowers his cost remains accepted.*
2. *The payment rule applies threshold payments, i.e., every accepted buyer is charged his threshold value — the lowest value he could have reported while remaining accepted, and every accepted seller is paid his threshold cost — the highest cost he could have reported while remaining accepted.*

Note that threshold payments are sufficient to guarantee IR, and since all the mechanisms we consider apply threshold payment, we do not discuss individual rationality further.

A similarly simple characterization of WGSP and IR double auction mechanisms is not available.<sup>4</sup>

### 3. Composition Framework

In this section we describe our framework for designing double auctions via composition. We first describe the one-sided algorithms and then the different ways of composing them.

#### 3.1. Ranking Algorithms

The one-sided algorithms we use for our compositions are called *ranking algorithms*. A ranking algorithm for buyers (sellers) is a deterministic algorithm that receives as input a value profile  $\vec{v}$  (cost profile  $\vec{c}$ ), and returns an ordered set of buyers (sellers), which we refer to as a *stream*. Not all buyers (sellers) must appear in this stream, e.g., for feasibility considerations. The *rank* of a buyer (seller), denoted by  $r_i(\vec{v})$  ( $r_j(\vec{c})$ ), is his position in the stream (e.g., 1 if he appears first), or  $\infty$  if he does not appear in the stream. The closer a player's rank is to 1, the *higher* he is ranked. Accessing the next player in the stream is called *querying* the algorithm. When querying the  $k$ th player, the *history* consists of the identities and values/costs of the  $k - 1$  previously-queried players.

We distinguish between two natural feasibility properties, based on the feasibility set system of the relevant side of the market. A *forward-feasible* ranking algorithm returns a stream of players such that any prefix of the stream is

<sup>3</sup>A stronger notion of group strategyproofness requires that no group of buyers and sellers can jointly deviate to make some member of the group strictly better off while all other members are no worse off. This stronger notion is violated by all common double-auction formats. For example, if a seller's cost sets the price for a buyer, then the seller can claim to have a lower cost to lower the buyer's payment without affecting its own utility.

<sup>4</sup>See [22] for recent progress towards characterizing WGSP and BB mechanisms in the context of cost sharing mechanisms.

a feasible set. A *backward-feasible* ranking algorithm returns a stream of players such that there is a minimal prefix of the stream which must be *discarded* to get a feasible set, and such that discarding additional players maintains feasibility.

The semantic difference between forward-feasible and backward-feasible ranking algorithms is that the former returns a stream of players who can be greedily accepted for trade, while the latter returns a stream of players who can be greedily rejected. This difference can be important for the properties that we study.

*Running Examples.* Let us briefly recall some standard, greedy-based algorithms for the three running examples. These algorithms find an (approximately-) optimal feasible subset  $T$  of the ground set  $U$ , where elements of  $U$  are weighted by their qualities (values/costs). It is straightforward to then turn these into forward- resp. backward-feasible ranking algorithms.

1. *Matroids:* The greedy algorithm for finding an optimal independent set sorts the elements of the ground set by their weight, and then goes through the sorted list, adding the next element to  $T$  if doing so does not violate feasibility.
2. *Knapsacks:* Recall that computing the maximum weight solution is NP-hard. There is a standard FPTAS (fully polynomial time approximation scheme) based on rounding and dynamic programming. The standard greedy approximation algorithm adds elements to  $T$  by density while maintaining feasibility; this is a 2-approximation. An alternate greedy algorithm ranks elements by weight, and achieves an approximation ratio of  $((1 - \lambda)\mu)^{-1}$  (where recall  $\lambda \leq 1$  is the largest element to knapsack size ratio, and  $\mu \leq 1$  is the smallest to largest element ratio).
3. *Matchings:* The maximum weight solution can be computed in polynomial time; the greedy algorithm that sorts edges by their weight, and accepts the next edge if neither of its endpoints was previously added, achieves a 2-approximation.

### 3.2. Composition of Ranking Algorithms

A *composition* of two ranking algorithms is a feasible double auction mechanism, whose allocation rule iteratively queries the ranking algorithms for the next buyer and next seller, and decides whether to accept or reject them based on their value and cost and the current history, and whose payment rule applies threshold payments.

*Composition Rule.* The decision whether to accept or reject a buyer-seller pair is performed by a *composition rule*.

For example, a *t-threshold* composition rule accepts a buyer-seller pair  $(i, j)$  if and only if its gain from trade  $v_i - c_j$  exceeds  $t$ . A *lookback* composition rule decides whether to accept or reject a buyer-seller pair  $(i, j)$  without observing their value and cost  $v_i, c_j$ , but rather depending only on the history of values and costs of previously-queried players. A *lookback t-threshold* composition rule is a lookback rule that accepts a buyer-seller pair  $(i, j)$  if and only if the current history (or part of it) contains a previously-queried pair  $(i', j')$  whose gain from trade  $v_{i'} - c_{j'}$  exceeds  $t$ .

*Composition Direction.* We are interested in compositions which in addition to the composition rule also have a *direction*. A composition is *forward* if it composes two forward-feasible ranking algorithms as described in Algorithm 1.

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#### ALGORITHM 1: Forward Composition

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Input: Two forward-feasible ranking algorithms, one for the buyers and one for the sellers.

1. Given value and cost profiles  $\vec{v}, \vec{c}$ , query both ranking algorithms to form a buyer-seller pair  $(i, j)$ ; if one stream runs out of players go to (3).
  2. Based on  $(v_i, c_j)$  and the history of values and costs previously queried, use the composition rule to decide whether to accept buyer-seller pair  $(i, j)$  and go to (1), or go to (3).
  3. Stop, rejecting all remaining players.
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A composition is *backward* if it composes two backward-feasible ranking algorithms as described in Algorithm 2.

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**ALGORITHM 2:** Backward Composition

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Input: Two backward-feasible ranking algorithms, one for the buyers and one for the sellers.

0. Pre-processing: Given value and cost profiles  $\vec{v}, \vec{c}$ , query both ranking algorithms and reject players until the remaining sets  $B$  of buyers and  $S$  of sellers are feasible, and  $|B| = |S|$ .
  1. Query both ranking algorithms to form a buyer-seller pair  $(i, j)$ ; if the streams run out of players go to (3).
  2. Based on  $(v_i, c_j)$  and the history of values and costs previously queried in step (1) (excluding the pre-processing step), use the composition rule to decide whether to reject buyer-seller pair  $(i, j)$  and go to (1), or go to (3).
  3. Stop, accepting all remaining players.
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**Observation 3.1.** *A forward or backward composition is a feasible double auction mechanism.*

We remark that it is possible to implement some double auction mechanisms both as a forward composition and as a backward one. When this is the case we focus on the forward implementation.

#### 4. Incentives

This section presents our DSIC and WGSP composition theorems. We also discuss the implications of these theorems for our three running examples.

##### 4.1. DSIC Composition Theorem

A ranking algorithm is *rank monotone* if the rank of a player changes monotonically with his quality: A forward-feasible ranking algorithm for buyers is rank monotone if  $v_i < v'_i \Rightarrow r_i(\vec{v}) \geq r_i(v'_i, v_{-i})$  for every  $\vec{v}, i$ ; and a forward-feasible ranking algorithm for sellers is rank monotone if  $c_j < c'_j \Rightarrow r_j(\vec{c}) \leq r_j(c'_j, c_{-j})$  for every  $\vec{c}, j$ . For backward-feasible ranking algorithms, these are reversed:  $v_i < v'_i$  implies  $r_i(\vec{v}) \leq r_i(v'_i, v_{-i})$  and  $c_j < c'_j$  implies  $r_j(\vec{c}) \geq r_j(c'_j, c_{-j})$ .

A ranking algorithm is *consistent* if the (feasible) players are ordered by quality: A forward-feasible ranking algorithm for buyers is consistent if buyers are ranked by decreasing value; a forward-feasible ranking algorithm for sellers is consistent if sellers are ranked by increasing cost. For a backward-feasible ranking algorithm, let  $\ell$  denote the rank of the first player that does not have to be discarded for feasibility; a backward-feasible ranking algorithm is consistent if the players that have rank at least  $\geq \ell$  are ordered by quality.

A composition rule is *monotone* if for all buyer-seller pairs  $(i, j), (i', j')$ , and all histories  $h, h'$  where one history is a prefix of the other, if  $(i, j)$  is accepted given history  $h$  and  $v_i \geq v_i', c_j \leq c_j'$ , then  $(i', j')$  is accepted given history  $h'$ .<sup>5</sup> An example of a monotone composition rule is the  $t$ -threshold rule.

We are now ready to state our theorem; we state here the version for forward composition, but it applies equally well to backward composition, by an analogous argument.

**Theorem 4.1.** *A forward composition of consistent, rank monotone ranking algorithms using a monotone composition rule is a DSIC double auction mechanism.*

*Proof.* We apply the characterization of DSIC double auctions in Proposition 2.1 to show that the composition is DSIC. Since compositions apply threshold payments, we only need to show that the allocation rule is monotone.

Fix value and cost profiles  $\vec{v}, \vec{c}$ . We argue that an accepted buyer who raises his value remains accepted; a similar argument shows that an accepted seller who lowers his cost remains accepted, and this completes the proof of allocation monotonicity.

Denote the accepted buyer by  $i$ , and the seller with whom  $i$  trades by  $j$ . Let  $h$  be the history when the composition rule is applied to  $(i, j)$ . By rank monotonicity of the buyer ranking algorithm, if  $i$  raises his value then his rank weakly decreases. Let  $j'$  be the seller with which  $i$  is considered for trade after his rank decreases. Then  $c'_j \leq c_j$  by consistency of the seller ranking algorithm. Since the composition rule is monotone, and the history  $h'$  when it is applied to  $(i, j')$  is a prefix of  $h$ , the pair  $(i, j')$  must be accepted for trade as well.  $\square$

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<sup>5</sup>If the composition rule is used by a forward composition,  $h'$  is a prefix of  $h$ , and vice versa if it is used by a backward composition.



The following immediate corollary applies to VCG-style composition with the 0-threshold composition rule.

**Corollary 4.2.** *Every double auction that is obtained by forward composition using the  $t$ -threshold composition rule of two rank-monotone and consistent ranking algorithms is DSIC.*

Monotonicity of the ranking algorithms and composition rule is an expected requirement for a DSIC double auction, in light of the characterization in Proposition 2.1, and the examples in Appendix A show formally that it is necessary. The following example demonstrates why DSIC requires consistency of the ranking algorithms.

**Example 4.3.** Let  $n = m = 2$ . Consider a forward composition, by the 0-threshold composition rule, of a seller lexicographic ranking algorithm (seller 1 ranked before 2), with a buyer ranking algorithm by individual scoring functions. Let buyer 1's scoring function be  $v_1$ , and let buyer 2's scoring function be  $10v_2$ . Observe that the ranking algorithms are monotone but not consistent. For value and cost profiles  $\vec{v} = (20, 3)$  and  $\vec{c} = (10, 0)$ , buyer 2 has an incentive to misreport his value as  $0 \leq v'_2 < 2$ . This results in his ranking after buyer 1 instead of before him, and thus in his acceptance as a trading pair with seller 2 instead of rejection as a trading pair with seller 1. Moreover, he will be charged zero payment.

*Running Examples.* What implications does the DSIC composition theorem have for the three running examples? All ranking algorithms described in Section 3.1 except the FPTAS for Knapsack are rank monotone, and can easily be made consistent by sorting the elements of the (approximately) optimal feasible set  $T$  by their weight. Thus, the DSIC composition theorem and its corollary are relevant for all three examples.

#### 4.2. WGSP Composition Theorem

For our WGSP composition theorem we leverage the framework of *deferred-acceptance algorithms* [29], described in Algorithm 3. A *deferred-acceptance ranking algorithm* for buyers (sellers) runs a maximization (minimization) version of a deferred-acceptance algorithm to get a stream of buyers (sellers). A *deferred-acceptance auction* for sale (procurement) runs a maximization (minimization) deferred-acceptance algorithm to get the accepted players, and applies threshold payments.

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#### ALGORITHM 3: Deferred Acceptance Algorithm—Maximization and Minimization Versions

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Maintain a set  $A$  of *active* players — initially all players.

While  $A$  is nonempty:

- Scoring functions assign every active player  $i \in A$  a non-negative score  $s_i^A(b_i, b_{-A})$ , where  $b_i$  is the bid of player  $i$  and  $b_{-A}$  are the bids of the inactive players. The scoring functions are weakly increasing in the first argument.  
[A player's score is a function of his own bid, the bids of the inactive players, and the set of active players. The scoring function cannot depend on the *bids* of other active players.]
- In the *maximization (minimization)* version, the player with the lowest finite (highest nonzero) score is removed from the set of active players  $A$ . If all scores are  $\infty$  (0), exit.

The output is a stream of *rejected* players, in the order in which they were removed from  $A$ ; and the *accepted* players, who remain in  $A$ .

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**Theorem 4.4.** *A backward composition of deferred-acceptance ranking algorithms using a lookback composition rule is a WGSP double auction mechanism.*

*Proof.* We show the composition is equivalent to a deferred-acceptance auction (for sale); the theorem then follows from the results of [29], by which every deferred-acceptance auction is WGSP.

We begin by transforming the deferred-acceptance seller ranking algorithm from a minimization to a maximization version. This is done by defining new scoring functions as follows. Every new scoring function  $s_j^S(b_j, b_{-S})$  multiplies its first argument by  $-1$ , applies the original scoring function for seller  $j$  and active set  $S$ , multiplies the result again by  $-1$ , and adds a large enough constant to make the final score positive; if the original scoring function returns zero then  $s_j^S(b_j, b_{-S})$  outputs  $\infty$ . The new scoring functions are weakly increasing in the first argument, and if the sellers' costs

are interpreted as their *values* for being accepted—by multiplying the costs by  $-1$ —then the maximization version with the new scoring functions is equivalent to the minimization version with the original ones.

Our goal is now to use the original scores  $s_i^B(v_i, v_{-B})$  for buyers, and the new scores  $s_j^S(-c_j, c_{-S})$  for sellers, to define scoring functions for the deferred-acceptance implementation of the backward composition in Algorithm 2. If the set of active buyers  $B$  is infeasible, or if the set of active sellers  $S$  is such that  $|B| > |S|$ , we set the scores of all active buyers to  $s_i^B(v_i, v_{-B})$ , and the scores of all active sellers to  $\infty$ . This means that the player who will be rejected next is the buyer with the lowest score. This corresponds to step (0) of the backward composition as described in Algorithm 2. We proceed similarly if the set of active buyers  $B$  is feasible and  $|B| < |S|$ , or if the set of active sellers  $S$  is infeasible.

The interesting case is when the set of active buyers  $B$  is feasible, the set of active sellers  $S$  is feasible, and  $|B| = |S|$ . We are therefore in step (2), and must set the scoring functions to implement the decision of the lookback composition rule to accept or reject. Here we use the fact that the lookback composition rule’s decision is only allowed to depend on the values and costs of previously-rejected players. Since the scoring functions have access to the reports of all inactive players, they can distinguish between players queried in step (1) from players rejected in step (0) (by simulating the backward composition), and thus can implement any lookback composition rule. If the rule says that the current buyer-seller pair should be rejected, we set the scores of all active buyers to  $s_i^B(v_i, v_{-B})$ , and the scores of all active sellers to  $\infty$ . This means that the player who will be rejected next is the buyer with the lowest score. If the rule says that the current buyer-seller pair should be accepted, we set the scores of all active players to  $\infty$ , causing the deferred acceptance implementation to stop and accept the remaining active players, in accordance to step (3) of the backward composition as described in Algorithm 2.

This completes the construction of the scoring functions, providing a deferred-acceptance implementation of the composed double auction mechanism.  $\square$

We obtain the following corollary to the preceding theorem that applies to trade reduction-style composition with the lookback 0-threshold rule.

**Corollary 4.5.** *Every double auction that is obtained by backward composition using the lookback  $t$ -threshold composition rule of two deferred-acceptance ranking algorithms is WGSP.*

Two further corollaries for backward compositions of deferred-acceptance ranking algorithms using a lookback rule are: (1) Such a double auction can be implemented as a clock auction (from Proposition 13 in [29]); (2) In the double auction that uses the same allocation rule but charges first price payments, then there exists a Nash equilibrium in which the allocation and payments are identical to the DSIC outcome of the double auction with threshold payments (from Proposition 20 in [29]).

None of the strong properties shown in this section are shared by forward compositions (for counterexamples see Appendix B).

*Running Examples.* As in the case of the DSIC composition theorem, let us see what the implications of the WGSP composition theorem are for each of the three running examples. The greedy algorithm for matroids can be implemented as a deferred-acceptance algorithm (see Appendix C for details). The FPTAS for Knapsack cannot (otherwise it would have been WGSP and therefore also DSIC), but the greedy by weight algorithm can (see Appendix D for details). For matchings, neither the optimal nor the greedy by weight algorithm can be implemented via deferred acceptance; we therefore design a new deferred-acceptance algorithm for matchings (see Appendix E for details).

## 5. Welfare

In this section we discuss the welfare guarantees of double auction mechanisms arising from compositions, and the implications for the three running examples.

Let  $\text{OPT}$  denote the optimal welfare subject to feasibility that can be achieved by all players  $N, M$ ; let  $t^*$  be the number of trades in some solution that achieves this welfare. Now consider forward-feasible (backward-feasible) consistent ranking algorithms. Denote by  $t'$  an optimal number of trades subject to feasibility for a composition of these ranking algorithms, and by  $W$  the welfare achieved by accepting the  $t'$  highest (lowest) ranking buyer-seller pairs. Note that  $t'$  need not be identical to  $t^*$ , for example, because the ranking algorithms are not optimal due to

computational hardness. Finally, consider a forward (backward) composition that accepts the  $t \leq t'$  highest (lowest) ranking buyer-seller pairs. An example of a composition that accepts strictly less than  $t'$  buyer-seller pairs is the trade reduction mechanism; another reason for accepting less than  $t'$  buyer-seller pairs would be to achieve a higher revenue per accepted buyer-seller pair. Clearly, any such composition guarantees a  $t/t'$  fraction of  $W$ . The main difficulty in obtaining a welfare composition theorem is in relating this quantity  $W$  to the optimal welfare  $OPT$ .

### 5.1. Warm-Up: Welfare Composition Theorem for Matroids

For matroids and their greedy ranking algorithms, consider the  $t'$  highest (lowest) ranking buyer-seller pairs. These maximize welfare subject to feasibility, not only for a composition of these ranking algorithms but in general. Hence in this case  $W = OPT$ .

**Proposition 5.1.** *A forward (backward) composition of the greedy ranking algorithms for matroids, using a composition rule that accepts the  $t \leq t'$  highest (lowest) ranking buyer-seller pairs, achieves welfare at least*

$$\frac{t}{t'} \cdot OPT.$$

### 5.2. General Welfare Composition Theorem

For more general settings, instead of relating  $W$  directly to  $OPT$ , we first use that  $W$  can only be higher than the welfare  $W'$  achieved by accepting the  $t^*$  optimal buyer-seller pairs given the ranking algorithms, and then we prove that  $W'$  is at least a certain fraction of  $OPT$ . For this we need two tools.

The first tool uses that the ranking algorithms are close to optimal at all times; this allows us to argue that the value of the buyers (cost of the sellers) in  $W'$  is close to the value of the buyers (cost of the sellers) in  $OPT$ : Denote by  $v_{OPT}(u)$  ( $c_{OPT}(u)$ ) the value (cost) of the welfare-maximizing feasible solution of at most  $u$  buyers (sellers). For a given forward-feasible ranking algorithm, denote by  $v_{ALG}(u)$  ( $c_{ALG}(u)$ ) the value (cost) achieved by greedily allocating to the first  $u$  buyers (sellers) in the output stream (if there are less than  $u$ , take their total value (cost)). For a given backward-feasible ranking algorithm, the definitions are the same except that the last  $u$  buyers (sellers) in the feasible part of the output stream are considered. A ranking algorithm for buyers is a *uniform  $\alpha$ -approximation* if for every value profile  $\vec{v}$  and every  $u \leq n$ ,

$$v_{ALG}(u) \geq \frac{1}{\alpha} \cdot v_{OPT}(u).$$

A ranking algorithm for sellers is a *uniform  $\beta$ -approximation* if for every cost profile  $\vec{c}$  and every  $u \leq m$ ,

$$c_{ALG}(u) \leq \beta \cdot c_{OPT}(u).$$

The second tool—a standard tool with mixed-sign objective functions [cf. 35]—measures how close the optimal solution  $OPT = v_{OPT}(t^*) - c_{OPT}(t^*)$  is to 0 via the parameter  $\gamma = v_{OPT}(t^*)/c_{OPT}(t^*)$ . Clearly,  $\gamma \geq 1$  because in the welfare-maximizing solution the total value is at least the total cost. For  $\gamma = 1$  we have  $OPT = 0$ ; hence we focus on the case where  $\gamma > 1$  below. Intuitively, the closer  $\gamma$  is to 1, the closer the optimal welfare is to 0 (the difference of two large values), and the harder it is to achieve a good relative approximation.

**Theorem 5.2.** *The forward (backward) composition of two consistent ranking algorithms that are uniform  $\alpha$ - and  $\beta$ -approximations, using a composition rule that accepts the  $t \leq t'$  highest (lowest) ranking buyer seller pairs, achieves welfare at least*

$$\frac{t}{t'} \cdot \frac{\frac{\gamma}{\alpha} - \beta}{\gamma - 1} \cdot OPT.$$

Note that if  $\alpha = \beta = 1$ , the second term in the approximation factor vanishes and we get the bound stated in Proposition 5.1. The bound degrades gracefully from this ideal case in all relevant parameters.

*Proof.* Our goal is to show that

$$v_{ALG}(t) - c_{ALG}(t) \geq \frac{t}{t'} \cdot \frac{\frac{\gamma}{\alpha} - \beta}{\gamma - 1} \cdot (v_{OPT}(t^*) - c_{OPT}(t^*)).$$

Since the double auction is composed of forward-feasible (backward-feasible) consistent ranking algorithms, we can number the buyers and sellers from the beginning (end) of the respective streams by  $1, 2, \dots$  such that  $v_1 \geq v_2 \geq \dots \geq v_{t'}$  and  $c_1 \leq c_2 \leq \dots \leq c_{t'}$ . Using this notation,

$$v_{\text{ALG}}(t) - c_{\text{ALG}}(t) = \sum_{i=1}^t (v_i - c_i) \quad \text{and} \quad v_{\text{ALG}}(t') - c_{\text{ALG}}(t') = \sum_{i=1}^{t'} (v_i - c_i).$$

Another implication of the fact that the double auction is composed of consistent ranking algorithms is that the gain from trade is non-increasing. That is,  $i < j$  implies  $v_i - c_i \geq v_j - c_j$ . Hence for all  $s$  such that  $t < s \leq t'$  we have  $v_s - c_s \leq \frac{1}{t} \sum_{i=1}^t (v_i - c_i)$ . It follows that

$$\begin{aligned} v_{\text{ALG}}(t) - c_{\text{ALG}}(t) &= \sum_{i=1}^{t'} (v_i - c_i) - \sum_{i=t+1}^{t'} (v_i - c_i) \\ &\geq \sum_{i=1}^{t'} (v_i - c_i) - (t' - t) \frac{1}{t} \sum_{i=1}^t (v_i - c_i) \\ &= (v_{\text{ALG}}(t') - c_{\text{ALG}}(t')) - \left( \frac{t'}{t} - 1 \right) (v_{\text{ALG}}(t) - c_{\text{ALG}}(t)). \end{aligned}$$

Rearranging this shows

$$v_{\text{ALG}}(t) - c_{\text{ALG}}(t) \geq \frac{t}{t'} (v_{\text{ALG}}(t') - c_{\text{ALG}}(t')). \quad (1)$$

Recall that  $t^*$  is defined as the number of trades in a solution that maximizes welfare, while  $t'$  is the number of trades that maximizes welfare for the given ranking algorithms. By the definition of  $t'$  all trades up to and including  $t'$  are beneficial, while all subsequent trades yield a deficit. That is,  $v_s - c_s \geq 0$  for  $s \leq t'$  and  $v_s - c_s < 0$  for  $s > t'$ . Hence,

$$v_{\text{ALG}}(t') - c_{\text{ALG}}(t') \geq v_{\text{ALG}}(t^*) - c_{\text{ALG}}(t^*). \quad (2)$$

Finally, we use that the ranking algorithms are uniform  $\alpha$ - and  $\beta$ -approximations and the definition of  $\gamma$  to deduce that

$$\begin{aligned} v_{\text{ALG}}(t^*) - c_{\text{ALG}}(t^*) &\geq \frac{1}{\alpha} v_{\text{OPT}}(t^*) - \beta c_{\text{OPT}}(t^*) \\ &= \left( \frac{\gamma}{\alpha} - \beta \right) c_{\text{OPT}}(t^*) \\ &= \frac{\frac{\gamma}{\alpha} - \beta}{\gamma - 1} (v_{\text{OPT}}(t^*) - c_{\text{OPT}}(t^*)). \end{aligned} \quad (3)$$

Combining inequalities (1)–(3) completes the proof.  $\square$

We obtain the following corollaries for VCG- and trade reduction style-mechanisms with the 0-threshold or the lookback 0-threshold composition rule.

**Corollary 5.3.** *Consider the forward composition of two consistent ranking algorithms that are uniform  $\alpha$ - and  $\beta$ -approximations. The 0-threshold rule accepts the  $t'$  highest ranking buyer-seller pairs. Hence its approximation ratio is at least*

$$\frac{\frac{\gamma}{\alpha} - \beta}{\gamma - 1}.$$

**Corollary 5.4.** *Consider the backward composition of two consistent ranking algorithms that are uniform  $\alpha$ - and  $\beta$ -approximations. The the lookback 0-threshold rule accepts the  $t' - 1$  lowest ranking buyer seller pairs. Hence its approximation ratio is at least*

$$\left( 1 - \frac{1}{t'} \right) \left( \frac{\frac{\gamma}{\alpha} - \beta}{\gamma - 1} \right).$$

When  $\alpha = \beta = 1$ , these two corollaries specialize to the traditional guarantees of the VCG and trade reduction mechanisms. More generally, they provide gracefully degrading guarantees as  $\alpha$  and  $\beta$  exceed 1.

*Running Examples.* Since all ranking algorithms that we have not yet ruled out are consistent, it is the uniform approximation property that we have to check. The greedy algorithms for matroids are not only optimal, but also uniformly so (as we show in Appendix C). Similarly, the algorithm for knapsacks that ranks by weight is a uniform  $((1 - \lambda)\mu)^{-1}$  approximation (as we show in Appendix D). For matchings, we show that the algorithm that we propose is a uniform 2-approximation (see Appendix E).

## 6. Budget Balance

This section studies the budget-balance properties of compositions, and derives implications for the running examples.

### 6.1. Budget Balance Composition Theorem

We say that a backward composition *reduces an efficient trade* if there is a buyer-seller pair with non-negative gain from trade that is rejected in step (2) of Algorithm 2.

**Theorem 6.1.** *A backward composition of deferred-acceptance ranking algorithms using a lookback composition rule that reduces at least one efficient trade is a BB double auction mechanism.*

*Proof.* Assume without loss of generality that the buyer deferred-acceptance ranking algorithm returns the stream  $1, 2, \dots, n$  of buyers, and the seller deferred-acceptance ranking algorithm returns the stream  $1, 2, \dots, m$  of sellers. Let  $\ell_B$  ( $\ell_S$ ) be the rank of the first buyer (seller) who does not have to be rejected to obtain feasibility. Let the pair  $(\ell'_B, \ell'_S)$  be the ranks of the efficient buyer-seller pair that is reduced and let their value and cost be  $v^* \geq c^*$ . Observe that  $\ell'_B, \ell'_S > \max\{\ell_B, \ell_S\}$ .

Our goal is to show that every buyer whom the composition accepts pays at least the value  $v^*$ . Since the composition applies threshold payments, it is enough to show that every accepted buyer  $i > \ell'_B$  who lowers his value to  $v'_i < v^*$  will no longer be accepted. A symmetric argument shows that every seller whom the composition accepts is paid at most the cost  $c^*$ . This is sufficient to establish budget balance since  $v^* \geq c^*$ .

Let  $r'$  denote  $i$ 's rank by the buyer deferred-acceptance ranking algorithm after lowering his value to  $v'_i$ . Consider first the case  $r' < \ell_B$ . We now exploit a property of deferred-acceptance algorithms, namely that an accepted player can only alter the set of accepted players by becoming rejected. This is because the scores by which players are ranked depend only on their own value and on values of previously rejected players. By induction on the number iterations, the set of buyers ranked 1 to  $r' - 1$  does not change following the decrease in  $i$ 's value. We already know that rejecting buyers 1 to  $r' - 1$  is not enough to make the remaining buyers a feasible set, and so buyer  $i$  when ranked  $r'$  is necessarily rejected.

Consider now the case  $\ell_B \leq r'$ . By consistency of the buyer ranking algorithm, starting from rank  $\ell_B$  buyers are ranked in increasing order of value. It thus holds that  $\ell_B \leq r' < \ell'_B$ . As above, the set of buyers ranked 1 to  $r' - 1$  does not change following the decrease in  $i$ 's value. So by the fact that originally the pair ranked  $(\ell'_B, \ell'_S)$  was reduced, we know that given the history of values up to rank  $r' - 1$  and corresponding costs, the lookback composition rule rejects the next buyer-seller pair. We conclude that buyer  $i$  when ranked  $r'$  is necessarily rejected, completing the proof.  $\square$

Thus, when uniform 1-approximate ranking algorithms are available, the trade reduction mechanism is BB. For cases where either  $\alpha > 1$  or  $\beta > 1$ , Theorem 6.1 shows that suitable generalizations of this mechanism are BB.

*Running Examples.* Our backward composition theorem applies to all ranking algorithms that are implementable within the deferred-acceptance framework: the greedy algorithm for matroids, the greedy by weight algorithm for Knapsack, and the new matching algorithm that we describe in the appendix.

## 7. Lower Bounds

This section investigates the interplay between welfare, incentives and budget balance on the one hand and forward and backward composition on the other. We prove lower bounds on the welfare achievable by double auctions (compositions or not) that are either WGSP or DSIC and BB, and analyze the extent to which these can be achieved by double auctions resulting from forward or backward compositions.

### 7.1. Lower Bound Subject to WGSP

Our lower bound for WGSP mechanisms applies to anonymous mechanisms—mechanisms whose outcome does not change if the names of the players are permuted. Recall that  $t^*$  denotes the number of trades in the optimal feasible solution.

**Theorem 7.1.** *No anonymous double auction mechanism that is WGSP can achieve a worst-case approximation ratio strictly better than*

$$1 - \frac{1}{t^*}.$$

*Proof.* Assume by contradiction that there is an anonymous double auction mechanism that is WGSP and that, for some  $\epsilon > 0$ , achieves approximation ratio

$$1 - \frac{1}{t^*} + \epsilon.$$

Consider instances with  $t^*$  buyers with value  $v$  and  $t^*$  sellers with cost  $c$ . For all  $v > c$ , we get a contradiction to the claimed approximation ratio if strictly less than  $2t^*$  players trade. Hence, for all  $v > c$  all  $2t^*$  players must be winning. Similarly, for  $v < c$  we get a contradiction to the claimed approximation ratio if any players trade. For ease of presentation we assume that for  $v = c$  all  $2t^*$  players are winning.

Since the double auction mechanism is anonymous we know that winning buyers (sellers) with the same value (cost) must make (receive) identical payments. In particular, if all buyers (sellers) win and have the same value (cost) then all buyers (sellers) must make (receive) identical payments.

We claim that for all  $v \geq c > 0$  the double auction mechanism has to set the payments of the buyers  $p_B$  and the payments to the sellers  $p_S$  to

$$p_B = c \text{ and } p_S = v.$$

The arguments for the buyers and the sellers are symmetric, and so we only present the argument for the buyers.

We first show that the payments for buyers with values  $v = c$  must be  $p_B = c$ . If the payments are  $p_B > c$ , we get a contradiction to WGSP, because the buyers currently have utility  $v - p_B < 0$  and could jointly deviate to  $v' < c$  which would make them lose and pay nothing for a utility of zero. If the payments are  $p_B < c$ , then in an instance where buyers have values  $v'$  and sellers have costs  $c$  such that  $c > v' > p_B$ , the buyers could jointly deviate and report a value of  $c$ . Before the deviation they are losing and not paying anything for a utility of zero, after the deviation they are winning and paying  $p_B < c$  which gives them a strictly positive utility.

Next we show that the payments for buyers with values  $v > c$  must be exactly  $c$ . If the payments are  $p_B > c$ , then these buyers could strictly gain by a group deviation to  $c$ . This would strictly improve their utility from  $v - p_B$  to  $v - c$ , where we use that they pay exactly  $c$  if they report a value of  $c$ . If the payments are  $p_B < c$ , then in an instance where buyers have values  $v' = c$  and sellers have costs  $c$ , they could strictly gain by a group deviation to  $v$  because this will lower their payment from  $c$  to  $p_B$ .

The statement of the theorem follows from this partial characterization of the payments by considering an instance with  $t^*$  buyers and  $t^*$  sellers and a group deviation of all buyers and all sellers to  $v', c'$  such that  $v' > v \geq c > c'$ .  $\square$

From Corollary 5.4 we know that we can achieve the lower bound via the backward composition of uniformly 1-approximate, deferred-acceptance ranking algorithms with the lookback 0-threshold rule. We conclude that whenever the trade reduction mechanism can be implemented in this manner, it achieves optimal worst-case welfare subject to WGSP.

### 7.2. Lower Bound Subject to DSIC and BB

Next we show a lower bound that applies to all double auction mechanisms resulting from composition or not that are DSIC and BB. In Appendix F we prove a similar lower bound for double auctions that are the result of composition.

**Theorem 7.2.** *No double auction mechanism that is DSIC and BB can achieve a worst-case approximation ratio strictly better than*

$$1 - \frac{1}{t^*}.$$

*Proof.* For contradiction, assume that there is a DSIC and BB double auction that achieves an approximation ratio of

$$(1 + \epsilon) \left(1 - \frac{1}{t^*}\right)$$

for some  $\epsilon > 0$ .

Consider an instance with  $t^*$  buyers and  $t^*$  sellers. The buyers have a value of  $y$ , and the sellers have a cost of  $x < y$ .

Consider a unilateral deviation by some buyer to  $y'$  such that  $y \geq y' \geq x$ . We claim that the buyer must remain winning as long as

$$y' \geq \frac{\left(1 - \frac{1}{t^*} - \epsilon(t^* - 1)\right)y + \epsilon t^* x}{1 - \frac{1}{t^*} + \epsilon}. \quad (4)$$

To see this note that  $\text{OPT} = (t^* - 1)(y - x) + (y' - x)$ . If the buyer does not win, then the welfare achieved by the double auction is at most  $(t^* - 1)(y - x)$ . Together with inequality (4), this would violate the claimed approximation ratio.

Consider a unilateral deviation by some seller to  $x'$  such that  $y \geq x' \geq x$ . An analogous argument shows that the buyer must remain winning as long as

$$x' \leq \frac{\left(1 - \frac{1}{t^*} - \epsilon(t^* - 1)\right)x + \epsilon t^* y}{1 - \frac{1}{t^*} + \epsilon}. \quad (5)$$

Together with the DSIC requirement these arguments show that the payments of the buyers are at most the RHS of inequality (4) and the payments to the sellers are at least the RHS of inequality (5). We get a contradiction to BB if the former is smaller than the latter. This is the case for

$$\epsilon > \frac{t^* - 1}{2(t^*)^2 - t^*},$$

which we can ensure by choosing  $t^*$  sufficiently large. □

From Corollary 5.4 we again know that we can achieve this lower bound via the backward composition of uniformly 1-approximate, deferred-acceptance ranking algorithms with the lookback 0-threshold rule. Hence whenever the trade reduction mechanism can be implemented in this manner, it is not only worst-case optimal subject to WGSP but also subject to DSIC and BB.

### 7.3. Impossibility Result for Forward Composition

We conclude with an impossibility result, which shows that DSIC double auction mechanisms that are the result of forward composition are particularly ill-equipped to achieve either WGSP or budget balance while maintaining a non-trivial efficiency guarantee.

**Proposition 7.3.** *Consider a double auction setting with  $n = m = 2$  and no feasibility constraints. For every forward composition of consistent ranking algorithms that is DSIC, there exist value and cost profiles for which either the budget deficit is arbitrarily high and the mechanism is not WGSP, or the welfare is arbitrarily small with respect to OPT.*

*Proof.* Let  $H$  be an arbitrarily large constant. We show there exist value and cost profiles such that either the budget deficit is at least  $H/8$  and the mechanism is not WGSP, or the welfare is at most an  $8/H$ -fraction of the welfare achievable by the trade reduction double auction.

We define the following value and cost profiles:

$$\begin{aligned} \vec{v}^1 &= (H, H) & \vec{v}^2 &= (\frac{H}{4}, \epsilon) & \vec{v}^3 &= (H, \epsilon) & \vec{v}^4 &= (\frac{3H}{8}, \epsilon) \\ \vec{c}^1 &= (\frac{3H}{4}, H - \epsilon) & \vec{c}^2 &= (0, 0) & \vec{c}^3 &= (\frac{5H}{8}, H - \epsilon) & \vec{c}^4 &= (0, H - \epsilon). \end{aligned}$$

Observe that for profile pairs  $(\vec{v}^1, \vec{c}^1)$ ,  $(\vec{v}^2, \vec{c}^2)$ , the trade reduction double auction with greedy ranking algorithms achieves welfare of at least  $H/4$ , and for the profile pair  $(\vec{v}^4, \vec{c}^3)$  its welfare is zero. It is also both BB and WGSP for all profile pairs.

We first show that for all the above value (cost) profiles, we can assume that the ranking algorithms rank first the buyer (seller) with higher value (lower cost). This holds trivially for  $\vec{v}^1$  and  $\vec{c}^2$  given that the ranking algorithms have non-empty outputs (otherwise, the welfare is 0 and we are done). The other profiles are all of the form  $\vec{v} = (v_h, v_\ell)$  where  $v_h - v_\ell \geq H/4 - \epsilon$  and  $v_\ell \geq \epsilon$ ; and  $\vec{c} = (c_\ell, c_h)$  where  $c_h - c_\ell \geq H/4 - \epsilon$  and  $c_h \leq H - \epsilon$ . If the buyer ranking algorithm given  $\vec{v}$  does not rank first the higher buyer, by consistency it does not rank this buyer at all, and so the welfare can be arbitrarily small with respect to the welfare of the trade reduction double auction (e.g., when  $\vec{v}$  is paired with cost profile  $(v_\ell - \epsilon, v_\ell - \epsilon)$ ). The argument for the seller ranking algorithm is similar (e.g., when  $\vec{c}$  is paired with value profile  $(c_h + \epsilon, c_h + \epsilon)$ ).

Now consider profile pairs  $(\vec{v}^1, \vec{c}^1)$  and  $(\vec{v}^2, \vec{c}^2)$ . Let  $(v, c)$  be the value and cost of the first buyer-seller pair that the composition rule considers; observe in both cases it has either  $v = H$  or  $c = 0$ . By DSIC, the composition rule for the first buyer-seller pair is equivalent to setting a threshold  $t_B = t_B(c)$  on the buyer's value, and a threshold  $t_S = t_S(v)$  on the seller's cost. What are the possible thresholds  $t_B(0), t_S(H)$ ? If either  $t_B(0) > H/4$  or  $t_S(H) < 3H/4$  then the first buyer-seller pair is rejected, and the maximum welfare from the second buyer-seller pair is  $\epsilon$ , completing the proof. It is left to reason about the case in which  $t_B(0) \leq H/4$  and  $t_S(H) \geq 3H/4$ . We now show that if this is the case then there is a large budget deficit for the profile pair  $(\vec{v}^3, \vec{c}^2)$ , and in addition the WGSP property is violated.

Given  $(\vec{v}^3, \vec{c}^2)$ , the first buyer-seller pair that the composition rule considers has value and cost  $(H, 0)$ , which clears both thresholds and is accepted for trade. Threshold payments imply a deficit of  $\geq H/2$ , and since the most that the second buyer-seller pair can contribute to covering this deficit is  $\epsilon$ , the total budget deficit is  $\gg H/8$  for small enough  $\epsilon$ .

We conclude by showing a violation of WGSP. Consider the profile pair  $(\vec{v}^4, \vec{c}^3)$ ; the first buyer-seller pair has value and cost  $(3H/8, 5H/8)$ . If it is accepted then the welfare is negative and the proof is complete. Otherwise, consider a group deviation to the profile pair  $(\vec{v}^3, \vec{c}^4)$ . The first buyer-seller pair then has reported value and cost  $(H, 0)$  and is accepted with payments  $t_B(0) \leq 2H/8, t_S(H) \geq 6H/8$ . This deviation is strictly preferable to both players in the deviating pair, completing the proof.  $\square$

## 8. Conclusion and Discussion

Motivated by the complexity of double auction design, we proposed a modular approach to the design of double auctions that decomposes the design task into the task of designing greedy algorithms for either side of the market and a composition rule. Focusing on the unit-demand and unit-supply case, we proved a number of composition theorems for (approximate) efficiency, DSIC or WGSP, and BB, which relate the properties of the double auction to the properties of the modules used in its construction.

We instantiated our approach for three different feasibility structures—matroids, knapsacks, and matchings. For matroids we show that both the VCG mechanism and a natural analog of McAfee's trade reduction mechanism can be implemented via composition. For the other settings we were able to obtain VCG- and trade reduction-style mechanisms with gracefully degrading efficiency guarantees. We also identified a sense in which our guarantees are the best possible, subject to strong incentive or budget-balance constraints.

Our approach can be extended in several different ways. A first direction is to consider other desirable properties, such as good pay-as-bid implementations or good revenue guarantees. A particularly interesting question here is to explore the trade-off between accepting fewer buyer-seller pairs for trade for a higher per-pair revenue, and accepting more buyer-seller pairs at a lower per-pair revenue.



Another direction is to retain the strategic and non-strategic properties considered here, but to consider more complex cross-market constraints. A potential starting point could be a setting with single-minded buyers who each want to buy from a certain set of sellers, each of whom produces a single unit of a unique good.

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## A. Necessity of Monotonicity for DSIC

The examples in this appendix complement Section 4.1 by showing that monotonicity of the ranking algorithms and the composition rule is necessary for DSIC.

**Example A.1.** Let  $n = m = 1$ . Consider a forward composition by the 0-threshold composition rule of a buyer ranking rule that greedily ranks buyers with values up to a cap  $c > 0$ , and a seller ranking rule that greedily ranks sellers. Let the buyer’s value be  $c$  and the seller’s cost be 0. Then the buyer would be better off reporting a value of  $c - \epsilon$ .

**Example A.2.** Let  $n = m = 1$ . Consider a forward composition of greedy ranking algorithms by a composition rule that accepts a pair  $(i, j)$  if and only if  $0 \leq v_i - v_j < d$ . Let the buyer’s value be  $d$  and the seller’s cost be 0. Then the buyer would be better off reporting a value of  $d - \epsilon$ .

## B. Counterexamples for Forward Composition

This appendix supplements Section 4.2 with counterexamples showing that none of the stronger incentive properties of deferred-acceptance implementations is shared by forward compositions. All examples are for the forward composition of two consistent and rank-monotone greedy-by-quality ranking algorithms, with the monotone 0-threshold composition rule.

**Example B.1** (Failure of WGSP). Consider a setting with one buyer with value 10 and one seller with cost 5. Then both players win and the buyer pays 5, while the seller is paid 10. The buyer and seller could report 11 and 4 instead, which would decrease/increase their respective payments to 4 and 11.

**Example B.2** (Failure of implementability as a clock auction). Consider a setting with three buyers  $N = \{1, 2, 3\}$  with feasible sets  $\mathcal{I}_N = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$  and sellers  $M = \{1, 2, 3\}$  with feasible sets  $\mathcal{I}_M = 2^M$ . Suppose the buyer values are all positive, while the seller costs are zero. The outcome includes buyers  $\{1, 2\}$  if  $\max\{v_1, v_2\} > v_3$  and buyer  $\{3\}$  otherwise. This cannot be implemented by a deferred-acceptance algorithm, and hence by Proposition 14 in [29] it cannot be implemented as a clock auction.

**Example B.3** (Failure of first price equivalence). Consider a setting with one buyer with value 10 and one seller with cost 5. Then both players win and the buyer pays 5, while the seller is paid 10. To achieve the same payments in a first price version the bids must be 5 and 10, but then the players do not win.

## C. Ranking Algorithms for Matroids

In this appendix we present additional details for the greedy algorithm for matroids which we introduced in Section 3.1 and referred to in Sections 4.2 and 5.1. We first show that the greedy algorithm is a uniform 1-approximation; afterwards we show how to implement it within the deferred-acceptance framework.

**Proposition C.1.** *The ranking algorithms for matroids based on the greedy one-sided algorithm are uniform 1-approximations.*

*Proof.* That the greedy algorithm finds a maximum weight basis of any matroid is a well known fact [15]. The claim about the uniform approximation guarantee follows from the fact that if we restrict the independent sets to sets of size at most  $k$ , then the matroid structure is preserved [e.g., 39].  $\square$

**Proposition C.2.** *The backward-feasible ranking algorithm for matroids based on the greedy one-sided algorithm can be implemented as a deferred-acceptance algorithm.*

*Proof.* The greedy algorithm adds an element with highest weight that does not violate the feasibility constraint. This can be turned around by rejecting the element with lowest weight that forms a minimal dependent set with the unrejected elements [see also 5, 29]. This property is a structural property of the unrejected elements, i.e., no knowledge of their weights is required. The reverse greedy algorithm can therefore be implemented as a deferred-acceptance algorithm, using the following scores for active players in  $A$ :

$$s_i^A(b_i, b_{-A}) = \begin{cases} b_i & \text{if } i \text{ forms a minimal dependent set with a subset of } A \setminus \{i\}, \\ \infty & \text{otherwise.} \end{cases}$$

In the ranking algorithm, once enough players have been rejected so that the active set  $A$  becomes a basis, we can continue to score elements by weight in order to maintain consistency.  $\square$

#### D. Ranking Algorithms for Knapsacks

This appendix is devoted to the greedy by weight algorithm for Knapsack which we introduced in Section 3.1 and referred to in Sections 4.2 and 5.1. It gives intuition why this algorithm is a uniform  $1/((1-\lambda)\mu)$ -approximation, where  $\lambda \leq 1$  is the largest element to knapsack size ratio and  $\mu \leq 1$  is the smallest to largest element ratio. It also shows that this algorithm can be implemented as a deferred-acceptance algorithm.

**Proposition D.1.** *The ranking algorithms for knapsacks based on the greedy by weight one-sided algorithm are uniform  $\frac{1}{(1-\lambda)\mu}$ -approximations.*

*Proof sketch.* Recall that  $\lambda \leq 1$  is the ratio between the size of the largest element and the size of the knapsack, and  $\mu \leq 1$  is the ratio between the size of the smallest element and the size of the largest element. For every fixed cardinality  $k$ , there are two sources of optimality loss in the greedy by weight algorithm: (1) Elements of different size have approximately the same weight and the algorithm packs large instead of small elements. Loss of this kind can contribute up to a factor of  $1/\mu$  to the total loss. (2) A fraction of the knapsack effectively remains empty (can only be packed with worthless items). Loss of this kind can contribute up to a factor of  $1/(1-\lambda)$  to the total loss. The total loss is thus upper bounded by  $\frac{1}{(1-\lambda)\mu}$ , as required.  $\square$

**Proposition D.2.** *The backward-feasible ranking algorithm for knapsacks based on the greedy by weight one-sided algorithm can be implemented as a deferred-acceptance algorithm.*

*Proof.* The following scores can be used to compute an approximately maximum weight feasible set by a deferred-acceptance algorithm:

$$s_i^A(b_i, b_{-A}) = \begin{cases} b_i & \text{if } \sum_{j=1}^i \text{size}(i) > C \\ \infty & \text{otherwise.} \end{cases}$$

In the ranking algorithm, once enough players have been rejected so that the active set  $A$  becomes a basis, we can continue to score elements by weight in order to maintain consistency.  $\square$

#### E. Ranking Algorithms for Matchings

In this appendix we present the novel algorithm for matchings which we referred to in Sections 4.2 and 5.1. We describe it and analyze its approximation guarantee in Section E.1. Afterwards, in Section E.2, we show how to implement it within the deferred-acceptance framework.

### E.1. Backward-Feasible Ranking Algorithm

We describe a backward greedy algorithm that is based on an idea of Preis [33]. Our description (cf. Algorithm 4) follows that of Drake and Hougardy [13]. Unlike the algorithm of Drake and Hougardy our algorithm is randomized.

Our algorithm starts with an arbitrary node and then grows a path of locally heaviest edges. If such a path cannot be extended any further, it restarts this process at an arbitrary node. In the end — with probability  $1/2$  — it takes all even edges along the paths. Otherwise, it takes all odd edges. Once the remaining edges are feasible, we can continue to output edges in reverse order of their weight for the sake of consistency. This gives rise to a backward-feasible ranking algorithm.

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#### ALGORITHM 4: Path Growing Algorithm

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**Input:** Graph  $G = (V, E)$ , weights  $w(e) \geq 0$  for all edges  $e \in E$

**Output:** Matching  $M$

Set  $M_1 = \emptyset, M_2 = \emptyset, i = 1$ ;

**while**  $E \neq \emptyset$  **do**

Choose  $x \in V$  of degree at least 1 arbitrarily;

**while**  $x$  has a neighbor **do**

Let  $(x, y)$  be the heaviest edge incident to  $x$ ;

Add  $(x, y)$  to  $M_i$ ;

Set  $i = 3 - i$ ;

Remove  $x$  from  $G$ ;

Set  $x = y$ ;

**end**

**end**

Output  $M_1$  with probability  $1/2$ , otherwise output  $M_2$ ;

---

**Proposition E.1.** *The above backward-feasible ranking algorithm based on the path growing algorithm is consistent, rank monotone, and a uniform 2-approximation.*

*Proof.* The algorithm is consistent because it outputs the elements of the set  $M_i$  that has been picked in order of their weights.

It is rank monotone because by increasing its bid a player can only enter and not drop out of either  $M_1$  or  $M_2$ .

For the approximation guarantee we assign each edge to some node in the graph in the following way. Whenever a node is removed, all edges that are currently incident to that node are assigned to it. To prove the factor 2, we consider an optimal solution of cardinality  $k$ . Each of these edges is assigned to a node. If we consider the edges adjacent to these nodes that were added to  $M_1 \cup M_2$ , then from the fact that we picked the locally heaviest edges we know that their total weight is at least the weight of the optimal edges. The claim now follows from the fact that we pick each of these edges (or a better one) with probability  $1/2$ .  $\square$

### E.2. Implementation as Deferred Acceptance Algorithm

The randomization can be implemented by tossing a fair coin at the beginning of the algorithm, and by choosing  $M_1$  if the coin shows heads and by choosing  $M_2$  if the coin shows tails. The path growing part of the algorithm can be implemented as described in [14]. Once the set of active edges becomes feasible, we can continue to score by weight in order to maintain consistency.

## F. Additional Lower Bound Subject to DSIC and BB

This appendix complements Section 7.2 by proving a lower bound on the approximation ratio achievable by DSIC and BB double auction mechanisms that are the result of composition. Recall the definition of  $t'$  as the optimal number of trades subject to feasibility for a given composition of two ranking algorithms and of  $t^*$  as the number of trades in a welfare-maximizing solution. Also recall the definition of  $v_{\text{OPT}}(u)$  and  $c_{\text{OPT}}(u)$  as the value and cost of the optimal solution in which  $u$  buyers trade and the definition of  $\gamma = v_{\text{OPT}}(t^*)/c_{\text{OPT}}(t^*) \geq 1$ . As in the proof of Theorem 7.2 we focus on the case where  $\text{OPT} = v_{\text{OPT}}(t^*) - c_{\text{OPT}}(t^*) > 0$  and hence  $\gamma > 1$ .

**Theorem F.1.** Consider a double auction that is obtained by composing consistent, rank monotone ranking algorithms for the buyers and the sellers, which are uniform  $\alpha$ - and  $\beta$ -approximations. Then subject to DSIC and BB, the approximation ratio of this double auction cannot be better than

$$\left(1 - \frac{1}{t'}\right) \left(\frac{\gamma}{\alpha} - \beta\right).$$

To prove this theorem we need the following auxiliary lemma.

**Lemma F.2.** For every  $\epsilon \in (0, 1]$  and  $\frac{\gamma}{\alpha} > \beta \geq 1$ , there exists  $\delta > 0$  such that

$$(1 + \epsilon) \left(\frac{\gamma}{\alpha} - \beta\right) = (1 + \delta) \frac{\gamma}{\alpha} - \frac{1}{1 + \delta} \beta. \quad (\text{F.1})$$

*Proof.* Let  $\phi = \frac{\gamma}{\alpha\beta}$ . By assumption,  $\phi > 1$ . We divide Equation F.1 by  $(\frac{\gamma}{\alpha} - \beta)$  and substitute  $\phi$  to get

$$1 + \epsilon = \frac{(1 + \delta)\phi - \frac{1}{1 + \delta}}{\phi - 1}. \quad (\text{F.2})$$

Let  $h_\phi(\delta)$  be equal to the right-hand side of Equation F.2, i.e.,  $h_\phi(\delta) = ((1 + \delta)\phi - \frac{1}{1 + \delta})/(\phi - 1)$ . We now show that for every  $\phi \geq 1$ ,  $h_\phi(\delta)$  has the following properties:

1.  $h_\phi(0) = 1$ ;
2.  $h_\phi$  is continuous;
3.  $h_\phi$  is strictly increasing;
4. There exists  $\delta$  such that  $h_\phi(\delta) \geq 2$ .

Combining these properties completes the proof, since they imply that for every  $\phi > 1$  and every left-hand side  $1 + \epsilon \in (1, 2]$  of Equation F.2, there exists  $\delta > 0$  for which  $h_\phi(\delta)$  is equal to  $1 + \epsilon$ .

The proofs of properties (1) and (2) are straightforward. Property (3) follows since the derivative of  $h_\phi$  is

$$\frac{\phi + \frac{1}{(1 + \delta)^2}}{\phi - 1} > 0.$$

For property (4) substitute  $\delta = 1$ . □

We are now ready to prove the theorem.

*Proof of Theorem F.1.* For a contradiction assume that there is a double auction that is based on consistent, rank monotone ranking algorithms that are uniform  $\alpha$ - and  $\beta$ -approximations that is DSIC and BB and for some  $\epsilon > 0$  achieves an approximation ratio of

$$(1 + \epsilon) \left(1 - \frac{1}{t'}\right) \left(\frac{\gamma}{\alpha} - \beta\right).$$

Consider an instance with  $2t'$  buyers and  $2t'$  sellers. One half of the buyers has a value of  $y$  and the other half of the buyers has a value of  $y'$ . One half of the sellers has a cost of  $x$  and the other half has a cost of  $x'$ . Assume that  $\delta > 0$  and that the valuations and costs are as follows

$$\begin{aligned} y &> x \\ y' &= (1 + \delta) \frac{y}{\alpha} > x' = \frac{1}{1 + \delta} \beta x \end{aligned}$$

The approximation algorithms will only consider the part of the buyers and sellers with values  $y'$  and costs  $x'$ . Moreover, we will consider deviations from one of these buyers to  $x'' \leq y'' < y'$  and from one of these sellers to

$y' \geq x'' > x'$ . To ensure that picking only buyers and sellers from this group yields a  $\alpha$ - resp.  $\beta$ -approximation, it must be the case that

$$(t' - 1)y' + y'' \geq \frac{1}{\alpha}t'y, \text{ and} \quad (\text{F.3})$$

$$(t' - 1)x' + x'' \leq \beta t'x. \quad (\text{F.4})$$

To ensure that with a deviation to  $y''$  resp.  $x''$  the deviating buyer resp. seller still wins, we will argue that by only taking the remaining  $t' - 1$  buyer-seller pairs from that group will conflict with the approximation guarantee. That is, we will argue that in this case

$$\frac{ALG}{OPT} \leq \left(1 - \frac{1}{t'}\right) \frac{y' - x'}{y - x} < (1 + \epsilon) \left(1 - \frac{1}{t'}\right) \left(\frac{\frac{\gamma}{\alpha} - \beta}{\gamma - 1}\right). \quad (\text{F.5})$$

By Lemma F.2 we can ensure this by choosing  $\delta > 0$  so that for some  $0 < \epsilon' < \epsilon$ ,

$$y' - x' = (1 + \delta) \frac{y}{\alpha} - \frac{1}{1 + \delta} \beta x = \frac{(1 + \delta) \frac{\gamma}{\alpha} - \frac{\beta}{1 + \delta}}{\gamma - 1} (y - x) = (1 + \epsilon') \frac{\frac{\gamma}{\alpha} - \beta}{\gamma - 1} (y - x).$$

Using this and choosing  $y''$  and  $x''$  as small and as large as possible, without violating inequality (F.3) and inequality (F.4), we obtain

$$y'' - x'' = t' \left(\frac{1}{\alpha}y - \beta x\right) - (t' - 1)(y' - x') = (1 + \epsilon' - t'\epsilon) \left(\frac{\frac{\gamma}{\alpha} - \beta}{\gamma - 1}\right) (y - x).$$

For DSIC double auctions  $y''$  and  $x''$  are lower and upper bounds on the payments made by the buyers and received by the sellers. Hence choosing  $t'$  large enough so that the first part of the right-hand side and with it the entire right-hand side gets negative leads to the desired contradiction.  $\square$