Generalized Efficiency Bounds In Distributed Resource Allocation

Jason R. Marden Tim Roughgarden

***Abstract*—Game theory is emerging as a popular tool for distributed control of multiagent systems. To take advantage of these game theoretic tools, the interactions of the autonomous agents must be designed within a game-theoretic environment. A central component of this game-theoretic design is the assignment of a local utility function to each agent. One promising approach to utility design is assigning each agent a utility function according to the agent's *Shapley value*. This method frequently results in games that possess many desirable features, such as the existence and of pure Nash equilibria with near-optimal efficiency. In this paper, we explore the relationship between the Shapley value utility design and the resulting efficiency of both pure Nash equilibria and coarse correlated equilibria. To study this relationship, we introduce a simple class of resource allocation problems. Within this class, we derive an explicit relationship between the structure of the resource allocation problem and the efficiency of the resulting equilibria. Lastly, we derive a bicriteria bound for this class of resource allocation problems — a bound on the value of the optimal allocation relative to the value of an equilibrium allocation with additional agents.**

I. Introduction

Resource allocation is a fundamental problem that arises in many application domains ranging from the social sciences to engineering [2]–[7]. One example is the problem of routing information through a shared network, where the global objective is to minimize average delay [7]. An alternative example is the problem of allocating sensors to a given mission space where the global objective is to maximize coverage area [6]. Regardless of the specific application domain, the central objective is always the same: allocate resources to optimize a given global objective.

Research has focused on both centralized and distributed approaches for resource allocation [2], [5], [8]–[13]. In this paper, we study distributed algorithms for resource allocation in large-scale engineering systems, where a centralized control approach is undesirable or even infeasible. For example, a centralized control approach may be impossible for the aforementioned sensor allocation problem, due to the complexity associated with a potentially large number of sensors, the vastness/uncertainty of the mission space, or potential stealth requirements that restrict communication capabilities. A more desirable control approach is to establish a distributed control algorithm that allows the sensors to allocate themselves effectively over the mission space without the need for global intervention [14], [15]. Such an algorithm would eliminate the need for centralized communication and introduce an inherent robustness to communication failures, sensor failures, and environmental uncertainties. While desirable, establishing such a distributed control algorithm comes with its share of challenges. Is it possible to characterize the global behavior that results from the interactions of a large group of autonomous agents each acting independently in response to its own local information? How can we coordinate the agents behavior to ensure that the emergent global behavior is desirable? What do we give up in terms of efficiency when we transition from a centralized to a distributed control approach?

A popular tool for distributed resource allocation is game theory [11]–[13], [16]–[18]. Game theory is a well-established discipline in the social sciences used for describing the emergent global behavior in social systems such as traffic networks, social networks, and auctions. More generally, *"Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact"* [19]. The appeal of applying game-theoretic tools to distributed engineering systems stems from the fact that the *underlying* decision-making architecture in social systems and the *desired* decision-making architecture in distributed engineering systems can be analyzed using the same mathematical tools. Furthermore, the field of game theory provides a vast array of tools that are extremely valuable for the design and control of distributed engineering systems [20]–[22].

To take advantage of these game theoretic tools for distributed engineering systems, the interactions of the autonomous agents must be designed within a game-theoretic environment. This means that the system designer must specify the following elements: (i) the set of decision-making agents, (ii) a set of actions for each agent, and (iii) a local utility function for each agent. While specifying the agents and their respective actions can be relatively straightforward, assigning local utility functions is somewhat more opaque. There are many pertinent issues that need to be considered when designing the agents' utility functions including scalability, locality, tractability, and efficiency of the resulting stable solutions [16].

It is worth contrasting utility design with the well-developed field of economic mechanism design (e.g., [23]), which shares the goal of designing games in which self-interested behaviour leads to a desirable outcome. First, the primary challenge of economic mechanism design — the incentive-compatible elicitation of private preferences — is not relevant for utility design. For example, in a single-item auction, the willingness to pay of each bidder is a priori unknown, and the point of an auction is to determine who is willing to pay the most. By contrast, the point of utility design is to *define* the preferences of agents (i.e., programmable components) so that "self-interested" behavior in the resulting game leads to a good outcome. Second, the primary challenge of utility design for

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J. R. Marden is with the Department of Electrical, Computer, and Energy Engineering, University of Colorado, 425 UBC, Boulder, CO 80309, jason.marden@colorado.edu.

T. Roughgarden is with the Department of Computer Science, Stanford, 462 Gates Building, 353 Serra Mall, Stanford, CA 94305, tim@cs.stanford.edu.

decentralized systems — the lack of centralized control — is not relevant in traditional mechanism design. For example, in mechanism design it is usually assumed that the designer can centrally allocate resources and select the most favorable of many equilibria. These two fundamental differences render most techniques of mechanism design unsuitable for the decentralized control problems studied here.

To highlight the challenges inherent in utility design for distributed engineering systems, we now introduce the well-studied vehicle target assignment problem [24].

A. An Illustrative Example: The Vehicle Target Assignment Problem

The vehicle target assignment problem consists of a finite set of targets (or resources) denoted by R and each target (or resource) $r \in R$ has a relative worth $v_r \geq 0$ [24]. There are a finite number of vehicles (or agents) denoted by $N = \{1, 2, \ldots, n\}$. The set of possible assignments for vehicle i is $\mathcal{A}_i \subseteq 2^R$ and $\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$ represents the set of joint assignments. In general, the structure of $\mathcal{A}$ is not available to a system designer a priori. Lastly, each vehicle $i \in N$ is parameterized with an invariant success probability $1 \geq p_i \geq 0$ that indicates the probability vehicle i will successfully eliminate a target r given that $r \in a_i$. The benefit of a subset of agents $S \subseteq N$, $S \neq \emptyset$, being assigned to a target r is

$$W_r(S) = v_r \left(1 - \prod_{i \in S} [1 - p_i] \right), \tag{1}$$

where $\left(1 - \prod_{i \in S} [1 - p_i]\right)$ represents the joint probability of successfully eliminating target r. Accordingly, the goal of the vehicle target assignment problem is to find a joint assignment $a \in \mathcal{A}$ that maximizes the system level objective

$$W(a) = \sum_{r \in R : \{a\}_r \neq \emptyset} W_r\left(\{a\}_r\right), \tag{2}$$

where $\{a\}_r = \{i \in N : r \in a_i\}$.

In this paper we focus on the design of utility functions for the individual vehicles to ensure both the existence and the near-optimal efficiency of pure Nash equilibria.

One obvious choice is to assign each vehicle a utility function in accordance with the system level objective, i.e., we could define the utility of every vehicle $i \in N$ for an assignment $a \in \mathcal{A}$ as

$$U_i(a) = W(a). \tag{3}$$

This choice ensures that the optimal action profile is a pure Nash equilibrium. However, an agent cannot even evaluate its own utility unless it knows the entire assignment vector a. These severe informational demands on the individual agents preclude using such utility functions in the motivating application.

Accordingly, it is more desirable to assign each agent a utility function that depends only local information, meaning information pertaining to the targets that the individual vehicle is assigned to. The next obvious idea is to adapt system objective utility functions to satisfy this locality constraint. This can be done using the *marginal contribution utility* [16], [25], which has the following form: for every agent $i \in N$ and every allocation $a \in \mathcal{A}$

$$U_i^{\mathrm{MC}}(a) = \sum_{r \in a_i} \left(W_r\left(\{a\}_r\right) - W_r\left(\{a\}_r \setminus \{i\}\right) \right). \tag{4}$$

The marginal contribution utility also ensures that the optimal action profile is a pure Nash equilibrium. Unfortunately, there are generally additional Nash equilibria that are suboptimal. In a decentralized setting, it is not clear how to justify selecting one Nash equilibrium over another, and it is therefore ideal to have guarantees for *all* Nash equilibria.[1]

A second class of utility functions that satisfies these informational restrictions is the *Shapley value utility* [16], [30], [31], which has the following form: for every agent $i \in N$ and every allocation $a \in \mathcal{A}$

$$U_i^{\mathrm{SV}}(a) = \sum_{r \in a_i} \sum_{T \subseteq \{a\}_r \setminus \{i\}} \frac{|T|!(|S| - |T| - 1)!}{|S|!} \left(W_r(T \cup \{i\}) - W_r(T) \right). \tag{5}$$

The Shapley value utility also guarantees the existence of a pure Nash equilibrium; however, the optimal allocation is not guaranteed to be a pure Nash equilibrium as was the case with the marginal contribution utility. It is currently unresolved as to whether the Shapley value or marginal contribution utility provides better worst-case efficiency guarantees.

In this paper we focus on deriving efficiency bounds for the Shapley value utility for a simplified class of resource allocation problems described in Section I-B. Our motivation for considering the Shapley value utility, as opposed to the marginal contribution utility, is that the worst-case efficiency guarantees for the marginal contribution utility are often worse than those for the Shapley value utility (as we show in Example 2). Our results will characterize how the structure of the welfare functions impacts the efficiency of the resulting equilibria. Before formally stating our results, we introduce our model to make our contributions more clear.

B. Preliminaries: Model and Definitions

We consider the class of resource allocation problems where there is a set of agents $N = \{1, ..., n\}$ and a finite set of resources $R = \{r_1, ..., r_m\}$ that are to be shared by the agents. Each agent $i \in N$ is capable of selecting a single resource from a set $\mathcal{A}_i \subseteq R$. An action profile, or allocation, is represented by an action tuple $a = (a_1, a_2, ..., a_n) \in \mathcal{A}$ where the set of action profiles is denoted by $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. We restrict our attention to the class of *separable* and *anonymous* welfare functions of the form

$$W(a) = \sum_{r \in R} W_r\left(|a|_r\right)$$

where $|a|_r = |\{i \in N : a_i = r\}|$ is the number of agents that selected resource r in the allocation a and $W_r : \{0, 1, ..., n\} \to$

[1] There are distributed learning rules that often provide probabilistic convergence to these best case Nash equilibria, e.g., log linear learning [26]–[29]. However, the convergence rates associated with such rules is either uncharacterized or has been shown to be exponential in the game size in general resource allocation problems [26].

$\mathbb{R}^+$ is the anonymous welfare function for resource r; hence, the welfare generated at a particular resource depends only on the number of players using that resource. We restrict our attention to *submodular* welfare functions $\{W_r(\cdot)\}_{r\in R}$, i.e., each welfare function $W_r(\cdot)$ satisfies the following conditions: (i) positive, i.e., $W_r(k) \geq 0$ for all $k \geq 0$, (ii) non-decreasing, i.e., $W_r(k) \geq W_r(k-1)$ for all $k \geq 1$ and (iii) have decreasing marginal returns, i.e., $W_r(k)-W_r(k-1) \geq W_r(k+1)-W_r(k)$ for all $k \geq 1$.[2] In general a system designer would like to find an allocation that optimizes the system welfare

$$a^{\text{opt}} \in \arg\max_{a\in\mathcal{A}} W(a).$$

As highlighted above, we focus on game theory as a tool for obtaining distributed solutions to such resource allocation problems. We model the interactions of the agents as a strategic form game where the agent set is N, the action set of each agent is $\mathcal{A}_i$, and each agent is assigned the Shapley value utility function given in (5). In the case of anonymous welfare functions, the Shapley value utility simplifies to an equal share utility, i.e.,

$$U_i(a_i = r, a_{-i}) = \frac{1}{|a|_r} W_r(|a|_r) \tag{6}$$

where $a_{-i} = \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$ denotes the collection of action of all players other than player i. The Shapley value utility in (6) ensures that the resulting game is an instance of a congestion game; therefore, a pure Nash equilibrium is guaranteed to exist [34].[3] This is true irrespective of the agent set N, the resource set R, the action sets $\{\mathcal{A}_i\}_{i\in N}$, and the welfare functions $\{W_r\}_{r\in R}$. However, there are no general results characterizing the efficiency of the Nash equilibria that result from using the Shapley value utility design.

The focus of this paper is purely on understanding the efficiency of the resulting Nash equilibria when utilizing the Shapley value utility design in (6). To that end, define a *single selection anonymous resource allocation game with the Shapley value utility design* by the tuple $G = \{N, R, \{W_r\}_{r\in R}, \{\mathcal{A}_i\}_{i\in N}, \{U_i\}_{i\in N}\}$ and let $\mathcal{G}$ be the the entire class of such games. Note that we include $\{U_i\}_{i\in N}$ in the tuple even though the agents' utility functions are derived explicitly from the welfare functions $\{W_r\}_{r\in R}$ as given in (6). We use the worst case measure of the *price of anarchy (PoA)* to measure the efficiency of equilibria [35]. Informally, the price of anarchy provides an upper bound on the ratio between the performance of an optimal allocation a^{opt} and a Nash equilibrium a^{ne}. More formally, the price of anarchy is defined as

$$PoA = \sup_{G\in\mathcal{G}} \left(\max_{a^{\text{ne}}\in G} \frac{W(a^{\text{opt}}; G)}{W(a^{\text{ne}}; G)} \right)$$

Therefore, our definition implies that the price of anarchy will always be greater than or equal to 1. According to our definition, a 50% efficiency guarantee would correspond to a price of anarchy of 2. Defining the price of anarchy in this fashion will be convenient for the upcoming analysis.

C. Our Results

Our first result focuses on bounding the efficiency of pure Nash equilibria in the setting where the agents are *symmetric*. By symmetric, we mean that the action set of each agent is identical, i.e., $\mathcal{A}_i = \mathcal{A}_j$. Within this setting, in Theorem 2 we prove that given n agents the price of anarchy relative to pure Nash equilibria is of the form

$$\frac{W(a^{\text{opt}})}{W(a^{\text{ne}})} \leq 1 + \max_{r\in R,\ k\leq m\leq n} \left\{ \frac{W_r(k)}{W_r(m)} - \frac{k}{m} \right\}. \tag{7}$$

Accordingly, (7) provides a systematic methodology for computing a price of anarchy for a specific resource allocation problem by exploiting the structure of the objective functions W_r. As we will show in Example 1, this characterization will lead to significant improvements over the 50% efficiency guarantees presented in [16], [36].

Our second result focuses on bounding the efficiency of a broader class of equilibria, termed coarse correlated equilibria, in the setting where the agents are *asymmetric*. By asymmetric, we mean that the agents' action sets need not be identical, i.e., $\mathcal{A}_i \neq \mathcal{A}_j$. Here, a coarse correlated equilibrium is represented by a distribution over the joint action set, i.e., $z^{\text{cce}} \in \Delta(\mathcal{A})$ where $\Delta(\mathcal{A})$ denotes the simplex over the finite joint action set $\mathcal{A}$.[4] The performance associated with a coarse correlated equilibrium is taken as the expected value of the welfare. Obviously, by considering a broader setting with a broader set of equilibria the price of anarchy can only degrade over the characterization presented in (7). Within this setting, in Theorem 6 we prove that given n agents the price of anarchy relative to coarse correlated equilibria, i.e., a bound on the ratio $\frac{W(a^{\text{opt}})}{W(z^{\text{cce}})}$, is bounded above by

$$1+\max \left\{ \begin{array}{l} \max\limits_{r\in R,k\leq m\leq n} \left(\frac{W_r(k)}{W_r(m)} - \left(\frac{\max\{m+k-n,0\}+\min\{n-m,k\}\cdot\tilde{\beta}_r(m)}{m} \right) \right), \\ \max\limits_{r\in R,k\leq m\leq n} \left(1 - \left(\frac{\max\{k+m-n,0\}+\min\{n-m,k\}\cdot\tilde{\beta}_r(k)}{k} \right) \right), \end{array} \right. \tag{8}$$

where

$$\tilde{\beta}_r(m) = \frac{m}{m+1} \frac{W_r(m+1)}{W_r(m)}.$$

To highlight the implications of the characterizations presented in Theorems 2 and 6 we consider the following example.

Example 1 *Consider a resource allocation problem where each resource $r_j \in R$ has a submodular objective of the form $W_{r_j}(x) = x^{d_j}$ where $d_j \in [0,1]$. Without loss of generalities let $0 \leq d_1 \leq \ldots \leq d_n \leq 1$. Suppose each agent $i \in N$ is assigned a utility function according to the Shapley value as in (6). Figure 1 highlights the price of anarchy for both the symmetric and asymmetric settings as a function of d_1 where d_1 varies between $[0,1]$ when there are 100 agents. The price of anarchy is always bounded by 2; however, as*

[2] Submodularity corresponds to a notion of decreasing marginal returns and is a common feature of many system-level objective functions for engineering applications ranging from content distribution [32] to coverage problems [33].

[3] An action profile $a \in \mathcal{A}$ is a pure Nash equilibrium if for every agent $i \in N$ we have $U_i(a_i, a_{-i}) = \max_{a_i'\in\mathcal{A}_i} U_i(a_i', a_{-i})$. We will commonly express a pure Nash equilibrium as just a Nash equilibrium.

[4] We will formally define coarse correlated equilibria in Section III.

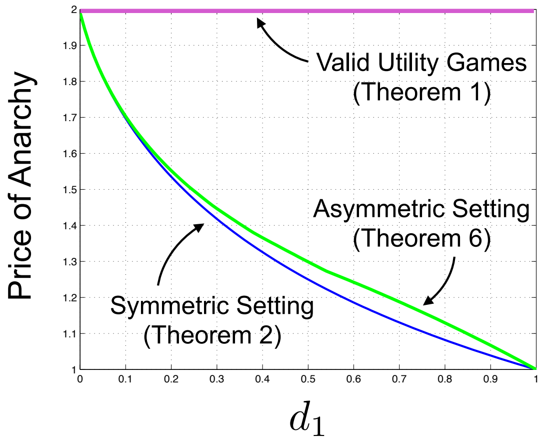

Fig. 1. Consider the resource allocation problem depicted in Example 1 where each resource $r_j \in R$ has a submodular objective of the form $W_{r_j}(x) = x^{d_j}$ where $d_j \in [0, 1]$ and each agent $i \in N$ is assigned a utility function in accordance with the Shapley value as in (6). Without loss of generality let $0 \leq d_1 \leq \ldots \leq d_n \leq 1$. This figure highlights the price of anarchy for both the symmetric and asymmetric settings as a function of d_1 where d_1 varies between $[0, 1]$. There are 100 agents. The price of anarchy is always bounded by 2; however, as $d_1 \to 1$, i.e., the objective functions become closer to linear, the price of anarchy approaches 1. Notice that the price of anarchy for the asymmetric setting is higher than the price of anarchy for the symmetric setting as expected.

$d_1 \to 1$, i.e., the objective functions become closer to linear, the price of anarchy approaches 1. Is is important to highlight that the values of $d_2, \ldots, d_n$ do not impact the price of anarchy guarantees. This example illustrates that the structure of the welfare function plays a prominent role in bounding the efficiency of either pure Nash equilibria or coarse correlated equilibria. Notice that the price of anarchy for the asymmetric setting is higher than the price of anarchy for the symmetric setting as expected.

A natural question is whether the Shapley value utility design provides better efficiency guarantees over the marginal contribution design. The following example illustrates that this is indeed the case for certain classes of resource allocation problems.

Example 2 *Consider the resource allocation problem depicted in Example 1 where the welfare function associate with each resource $r \in R$ is of the form $W_r(x) = c_r \cdot x^{0.5}$ where $c_r \geq 0$. The marginal contribution utility is of the form*

$$U_i^{\mathrm{MC}}(a_i = r, a_{-i}) = W_r(|a|_r) - W_r(|a|_r - 1).$$

Consider a specific resource allocation problem with player set $N = \{1, 2\}$, resource set $R = \{r_1, r_2, r_3\}$, action sets $\mathcal{A}_1 = \{r_1, r_2\}$ and $\mathcal{A}_2 = \{r_2, r_3\}$, and resource specific coefficients $c_{r_1} = c_{r_2} = 1$ and $c_{r_3} = \sqrt{2} - 1$. The optimal allocation for this setup is $a^{\mathrm{opt}} = (a_1 = r_1, a_2 = r_2)$ which yields a total welfare of 2. Under the marginal contribution utility, it is straightforward to verify that there exists a pure Nash equilibrium of the form $a^{\mathrm{ne}} = (a_1 = r_2, a_2 = r_3)$ which yields a total welfare of $\sqrt{2}$. Accordingly, for this specific example we have that $W(a^{\mathrm{opt}})/W(a^{\mathrm{ne}}) = 1.412$. Note that this exceeds the price of anarchy bound guarantees associated with the Shapley value for such scenarios which is approximately 1.3 (see Figure 1). Consequently, this proves that the Shapley value utility provides strictly better efficiency guarantees over the marginal contribution utility for this example. It is also important to highlight that this conclusion can be verified for all $d \in [0, 1]$ using this same example where $c_{r_3} = 2^d - 1$. It remains an open question as to whether the Shapley value utility provides stronger efficiency guarantees for all welfare functions.

Our second set of results focus on establishing bicriteria bounds, which we also refer to as the relative price of anarchy, for both the symmetric and asymmetric settings. By bicriteria bounds, we mean a bound on the value of the optimal allocation relative to the value of an equilibrium allocation with additional agents.

Bicriteria bounds are useful for weighing the relative cost/benefit of improving system behavior through (i) advancements in the underlying control design or (ii) introducing additional agents to the system. In particular, a bicriteria bound permits us to explore the tradeoff between the computational expense associated with more sophisticated dynamics for equilibrium selection, such as log-linear learning [27]–[29], and the physical cost associated with adding more agents to our system.

For the symmetric setting, in Theorem 3 we prove that given ln players in a pure Nash equilibrium and n players in a optimal allocation where $l \in \{1, 2, \ldots\}$, the relative price of anarchy (or bicriteria bound), i.e., a bound on the ratio $\frac{W(a^{\mathrm{opt}};n)}{W(a^{\mathrm{ne}};ln)}$, is bounded above by

$$\max\left\{1, \frac{1}{l} + \max_{r \in R,\ k \leq m \leq ln}\left(\frac{W_r(k)}{W_r(m)} - \frac{k}{m}\right)\right\}. \tag{9}$$

It is important to highlight that this bound is not tight and just provides an upper bound on the ratio $W(a^{\mathrm{opt}}; n)/W(a^{\mathrm{ne}}; ln)$. For the asymmetric setting, in Theorem 8 we prove that given ln players in a coarse correlated equilibrium and n players in a optimal allocation where $l \in \{1, 2, \ldots\}$, the relative price of anarchy (or bicriteria bound), i.e., a bound on the ratio $\frac{W(a^{\mathrm{opt}};n)}{W(z^{\mathrm{cce}};ln)}$, is bounded above by

$$1/l + \max\left\{\begin{array}{l}\max\limits_{r \in R, k \leq \min\{m_1, n\}, m_1 \leq ln}\left(\frac{W_r(k)}{W_r(m_1)} - \frac{k}{m_1+1}\frac{W_r(m_1+1)}{W_r(m_1)}\right), \\ \max\limits_{r \in R, m_2 \leq ln}\left(1 - \frac{m_2}{m_2+1}\frac{W_r(m_2+1)}{W_r(m_2)}\right).\end{array}\right. \tag{10}$$

Note that the bicriteria bound for pure Nash equilibria given in (9) is always greater than or equal to 1; hence, this bound cannot be used to determine the number of additional agents necessary for the Nash equilibrium associated with ln agents to offer a strict improvement over the performance of the optimal allocation with n agents. On the other hand, the bicriteria bound for coarse correlated equilibria given in (10) may attain values less than 1. Since the price of anarchy associated with coarse correlated equilibria must be greater than or equal to the price of anarchy associated with pure Nash equilibria, the bound given in (10) can be used to strengthen the bound given in (9) by providing a characterization of the price of anarchy when values are less than 1. Figure 2

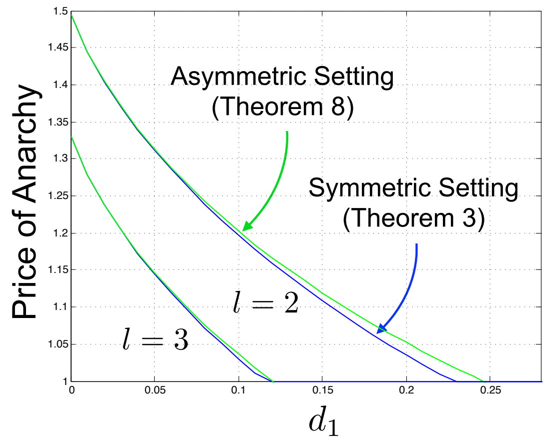

Fig. 2. Consider the resource allocation problem depicted in Example 1 where each resource $r_j \in R$ has a submodular objective of the form $W_{r_j}(x) = x^{d_j}$ where $d_j \in [0,1]$ and each agent $i \in N$ is assigned a utility function in accordance with the Shapley value. Without loss of generality let $0 \leq d_1 \leq ... \leq d_n \leq 1$. This figure highlights the relative price of anarchy for both the symmetric and asymmetric settings as a function of d_1, where d_1 varies between $[0,1]$. There are 100 agents when $l = 2$ and $l = 3$. The blue lines indicate the relative price of anarchy for pure Nash equilibria and symmetric agents given in Theorem 3 while the green lines indicate the relative price of anarchy for coarse correlated equilibria and asymmetric agents given in Theorem 8. Note that the relative price of anarchy associated with the symmetric setting is always less than the price of anarchy associated with the asymmetric setting as expected. However, the gap between the two bounds is relatively small. Furthermore, both bounds ensure that the performance associated with equilibrium behavior matches optimal behavior for certain ranges of d_1. For example, when $l = 2$, performance associated with a pure Nash equilibrium matches optimal behavior for $d_1 \geq 0.23$ while performance associated with a coarse correlated equilibrium matches optimal behavior for $d_1 \geq 0.25$.

illustrates these bounds on the resource allocation problem given in Example 1.

Lastly, in Section IV we conclude the paper with a detailed study of these bounds on the well-studied vehicle target assignment problem [24] which represents a special class of the resource allocation problems discussed in this paper.

II. The Symmetric Case

In this section, we focus on deriving efficiency bounds for pure Nash equilibria in the above class of resource allocation problems with symmetric agents. By symmetric, we mean that each agent's action set is identical, i.e., $\mathcal{A}_i = R$ for all $i \in N$. We consider asymmetric agents in Section III.

A. Review of Valid Utility Games

We begin by reviewing the results in [36] pertaining to *valid utility games*. A valid utility game is any game which satisfies the conditions set forth in the following theorem.

Theorem 1 (Vetta, 2002 [36]) *Consider a game with agents N, action sets $\{\mathcal{A}_i\}_{i\in N}$, utility functions $\{U_i\}_{i\in N}$, and a global objective $W : \mathcal{A} \to \mathbb{R}^+$. Suppose the following conditions are satisfied:*

(i) The global objective W is submodular.[5]

(ii) For every agent $i \in N$ and action profile $a \in \mathcal{A}$ we have $U_i(a) \geq W(a) - W(\emptyset, a_{-i})$ where $\emptyset$ represents the "null" action.[6]

(iii) For every action profile $a \in \mathcal{A}$ the agents' utility functions satisfy $\sum_{i\in N} U_i(a) \leq W(a)$.

Then if a pure Nash equilibrium exists the price of anarchy is 2.

B. Bounding the Efficiency of Pure Nash Equilibria

It is straightforward to show that every single selection anonymous resource allocation game with submodular resource-specific welfare functions and the Shapley value utility design satisfies Conditions (i)–(iii) in Theorem 1 [16]. Hence, Theorem 1 ensures that the price of anarchy is at most 2 irrespective of the number of resources or the number of players. The central question that we explore in this section is whether this price of anarchy of 2 is tight. It turns out that we can significantly sharpen this price of anarchy bound by taking into account the structure of the welfare function and the number of agents as shown in the following theorem.

Theorem 2 *Consider any anonymous single selection resource allocation game with n agents, symmetric action sets, the Shapley value utility design, and submodular welfare functions. The price of anarchy associated with pure Nash equilibria is bounded above by*

$$\frac{W(a^{\rm opt})}{W(a^{\rm ne})} \leq 1 + \max_{r\in R,\ k\leq m\leq n} \left\{ \frac{W_r(k)}{W_r(m)} - \frac{k}{m} \right\}. \quad (11)$$

Proof: Let $a^{\rm ne}$ and $a^{\rm opt}$ represent a Nash equilibrium and an optimal allocation respectively. For notational simplicity, define $|a^{\rm ne}| = \{x_1, ..., x_m\}$ and $|a^{\rm opt}| = \{y_1, ..., y_m\}$. Let z_r denote the number of agents that select resource r in both $a^{\rm ne}$ and $a^{\rm opt}$, i.e.,

$$z_r = \left|\{i \in N : a_i^{\rm ne} = r\} \cap \{i \in N : a_i^{\rm opt} = r\}\right|.$$

Note that $z_r \leq \min\{x_r, y_r\}$ for all resources $r \in R$.

We begin with a fairly generic derivation. First, we utilize the fact that the Shapley value utility is budget-balanced, i.e., $W(a) = \sum_{i\in N} U_i(a)$ for any action profile $a \in \mathcal{A}$, and the fact that $a^{\rm ne}$ is a Nash equilibrium to derive the following

$$\begin{aligned} W(a^{\rm ne}) &= \sum_{i\in N} U_i(a_i^{\rm ne}, a_{-i}^{\rm ne}), \\ &\geq \sum_{i\in N} U_i(a_i^{\rm opt}, a_{-i}^{\rm ne}), \\ &= \sum_{r\in R} z_r \frac{W_r(x_r)}{x_r} + (y_r - z_r)\frac{W_r(x_r+1)}{x_r+1}. \end{aligned} \quad (12)$$

[5]A global objective function W is submodular if for any action profile $a \in \mathcal{A}$, agent sets $S \subseteq T \subseteq N$, and agent $i \in N$ we have $W(a_{S\cup\{i\}}) - W(a_S) \geq W(a_{T\cup\{i\}}) - W(a_T)$ where $a_S = \{a_i : i \in S\}$. Here, we use the shorthand notation a_S to describe the action profile $a = (a_S, \{\emptyset\}_{j\notin S})$, i.e., all players $j \notin S$ selected the null action.

[6]Alternative, a player's utility is always greater than or equal to the player's marginal contribution to the global objective W.

Let R_1 and R_2 denote the resources on which $x_r \geq y_r$ and $x_r < y_r$, respectively. Focusing on the second set of terms in (12), we have that for any resource $r \in R_2$

$$\begin{aligned}(y_r - z_r)\,&\frac{W_r(x_r+1)}{x_r+1} \\ &= (x_r - z_r)\,\frac{W_r(x_r+1)}{x_r+1} + (y_r - x_r)\,\frac{W_r(x_r+1)}{x_r+1}, \\ &\geq (x_r - z_r)\,\frac{W_r(x_r+1)}{x_r+1} + (y_r - x_r)\left(W_r(x_r+1) - W_r(x_r)\right), \\ &\geq (x_r - z_r)\,\frac{W_r(x_r+1)}{x_r+1} + W_r(y_r) - W_r(x_r),\end{aligned}$$

where the second and third steps result from the discrete concavity of W_r. Therefore

$$\begin{aligned}\sum_{r\in R_2} &(y_r - z_r)\,\frac{W_r(x_r+1)}{x_r+1} \\ &\geq \sum_{r\in R_2}\left((x_r - z_r)\,\frac{W_r(x_r+1)}{x_r+1} + W_r(y_r) - W_r(x_r)\right), \\ &= W(a^{\rm opt}) - \sum_{r\in R_1} W_r(y_r) \\ &\qquad + \sum_{r\in R_2}\left((x_r - z_r)\,\frac{W_r(x_r+1)}{x_r+1} - W_r(x_r)\right),\end{aligned}$$

where the equality results from adding and subtracting $\sum_{r\in R_1} W_r(y_r)$. Plugging into (12) and simplifying gives us

$$\begin{aligned}W(a^{\rm ne}) \geq W(a^{\rm opt}) &+ \\ \sum_{r\in R_1} &\left(z_r\frac{W_r(x_r)}{x_r} + (y_r - z_r)\,\frac{W_r(x_r+1)}{x_r+1} - W_r(y_r)\right) + \\ \sum_{r\in R_2} &\left(z_r\frac{W_r(x_r)}{x_r} + (x_r - z_r)\,\frac{W_r(x_r+1)}{x_r+1} - W_r(x_r)\right). \quad (13)\end{aligned}$$

Since the players are symmetric, this means that each player $i \in N$ can select any resource $r \in R$. Therefore, permuting the players in the optimal allocation $a^{\rm opt}$ is well-defined and yields another optimal solution. Thus, for every pure Nash equilibrium $a^{\rm ne}$, we can choose the optimal allocation $a^{\rm opt}$ such that, for every resource $r \in R$, there is a set of $\min\{x_r, y_r\}$ players that use r in both $a^{\rm ne}$ and $a^{\rm opt}$. Accordingly, for every resource $r \in R_1$ we have $z_r = y_r$ and for every resource $r \in R_2$ we have $z_r = x_r$. Therefore, after the appropriate cancellations, (13) simplifies to

$$\begin{aligned}W(a^{\rm ne}) &\geq W(a^{\rm opt}) + \sum_{r\in R_1}\left(y_r\frac{W_r(x_r)}{x_r} - W_r(y_r)\right), \\ &= W(a^{\rm opt}) - \sum_{r\in R_1} W_r(x_r)\left(\frac{W_r(y_r)}{W_r(x_r)} - \frac{y_r}{x_r}\right).\end{aligned} \quad (14)$$

Define

$$\gamma(n) = \max_{r\in R,\ k\leq m\leq n}\left(\frac{W_r(k)}{W_r(m)} - \frac{k}{m}\right). \quad (15)$$

Substituting (15) into (14) gives us

$$\begin{aligned}W(a^{\rm ne}) &\geq W(a^{\rm opt}) - \gamma(n)\sum_{r\in R_1} W_r(x_r) \\ &\geq W(a^{\rm opt}) - \gamma(n)W(a^{\rm ne}).\end{aligned}$$

Finally, rearranging shows that

$$\frac{W(a^{\rm opt})}{W(a^{\rm ne})} \leq 1 + \gamma(n).$$

■

The value of Theorem 2 is that it provides a systematic method for establishing a price of anarchy by identifying the situation that gives rise to the worst efficiency given any set of submodular welfare functions. Regardless of the structural form of the welfare function, establishing a price of anarchy simplifies to a maximization of (11) over the parameter set $k \leq m \leq n$ and R.

Lastly, it is important to highlight that in general it is impossible to get a tighter bound than the bound presented in (11) as several welfare functions have examples which hit this bound. To see this, consider a resource allocation problem with resource set $R = \{r_1, r_2, \ldots, r_n\}$ where each resource has a scaled resource-specific welfare function of the form $W_r(S) = c_r \cdot \tilde{W}(S)$ where $c_r > 0$ is a scaling coefficient and $\tilde{W} : N \to \mathbb{R}$ is the "base" welfare function. Notice that the chosen constants c_r do not impact the price of anarchy bounds given in (11). Let

$$\{k^*, m^*\} \in \arg\max_{k\leq m\leq n}\left(\frac{\tilde{W}(k)}{\tilde{W}(m)} - \frac{k}{m}\right)$$

and suppose that n is divisible by m^*, which we require to show tightness. Denote the scaling coefficient associated with resource r_k as c_k. Let $c_k = 1$ for all $k \in \{1, \ldots, n/m^*\}$. For all $k \in \{n/m^* + 1, \ldots, n\}$, define the scaling coefficient to satisfy

$$c_k\left(\tilde{W}(1) - \tilde{W}(0)\right) = \frac{\tilde{W}(m^*)}{m^*}.$$

Here, a pure Nash equilibrium is characterized by m^* agents at each of the first n/m^* resources, i.e., all resources $r_k \in R$ where $c_k = 1$. The optimal allocation is characterized by k^* agents at each of the first n/m^* resources and the remaining agents are alone at $n - \frac{nk^*}{m^*}$ of the additional $n - \frac{n}{m^*}$ resources. Accordingly, the inefficiency of this equilibrium is

$$\begin{aligned}\frac{W(a^{\rm opt})}{W(a^{\rm ne})} &= \frac{\frac{n}{m^*}\tilde{W}(k^*) + \left(n - \frac{nk^*}{m^*}\right)\frac{\tilde{W}(m^*)}{m^*}}{\frac{n}{m^*}\tilde{W}(m^*)}, \\ &= 1 + \frac{\tilde{W}(k^*)}{\tilde{W}(m^*)} - \frac{k^*}{m^*},\end{aligned}$$

which precisely equals the derived price of anarchy bound in (11).

C. A Bicriteria Bound for Pure Nash Equilibria

In this section we explore the relative price of anarchy, or bicriteria bound, in single selection anonymous resource allocation games with submodular objective functions. By bicriteria bound, we mean a bound on the value of the optimal allocation relative to the value of an equilibrium allocation with additional agents. The following theorem establishes an explicit relationship between the price of anarchy and the number of additional agents present at the Nash allocation.

Theorem 3 *Consider any anonymous single selection resource allocation game with symmetric action sets, the Shalpey value utility design, and submodular welfare functions. The relative price of anarchy associated with any pure Nash equilibrium with ln players and an optimal allocation with n players where $l \in \{1, 2, \dots\}$ is bounded above by*

$$\frac{W(a^{\rm opt}; n)}{W(a^{\rm ne}; ln)} \leq \max\left\{1, \frac{1}{l} + \gamma(ln)\right\}. \quad (16)$$

Proof: Let $a^{\rm ne}$ and $a^{\rm opt}$ represent a pure Nash equilibrium and an optimal allocation respectively. For notational simplicity, define $|a^{\rm ne}| = \{x_1, ..., x_m\}$ and $|a^{\rm opt}| = \{y_1, ..., y_m\}$. Here, we have $\sum_{k=1}^m x_k = ln$ and $\sum_{k=1}^m y_k = n$. Let R_1 and R_2 denote the resources on which $x_r \geq y_r$ and $x_r < y_r$, respectively. Since the game is symmetric, we can define hypothetical deviations for the players from $a^{\rm ne}$ as follows:

(i) For each $r \in R_2$, x_r players deviate from r to itself, i.e., do not deviate at all.

(ii) For each $r \in R_2$, $(ly_r - x_r)$ players from resources in R_1 deviate to r.

(iii) The remaining players (on R_1) do not deviate.

Let $\tilde{a}$ be the resulting action profile after the above deviations and let g_r denote the number of such players utilizing resource $r \in R_1$ after the above deviations. Since $\sum_{r\in R} x_r = l\sum_{r\in R} y_r$ and ly_r players deviated to each resource $r \in R_2$, i.e., through (i) and (ii), we have $\sum_{r\in R_1} g_r = l\sum_{r\in R_1} y_r$. Since $g_r \leq x_r$ for each resource $r \in R_1$, we have

$$\begin{aligned}
&W(a^{\rm ne}; ln) \quad (17)\\
&= \sum_{i=1}^{ln} U_i(a^{\rm ne}),\\
&\geq \sum_{i=1}^{ln} U_i(\tilde{a}_i, a^{\rm ne}_{-i}),\\
&= \sum_{r\in R_1} g_r \frac{W_r(x_r)}{x_r} + \sum_{r\in R_2} x_r \frac{W_r(x_r)}{x_r}\\
&\qquad + \sum_{r\in R_2} (ly_r - x_r)\frac{W_r(x_r+1)}{x_r+1},\\
&= \sum_{r\in R_1} g_r \frac{W_r(x_r)}{x_r} + \sum_{r\in R_2} W_r(x_r) \quad (18)\\
&\qquad + \sum_{r\in R_2} (ly_r - x_r)\frac{W_r(x_r+1)}{x_r+1}.
\end{aligned}$$

Focusing on the third set of terms in (18) gives us

$$\begin{aligned}
&\sum_{r\in R_2} (ly_r - x_r)\frac{W_r(x_r+1)}{x_r+1}\\
&= \sum_{r\in R_2} (y_r - x_r)\frac{W_r(x_r+1)}{x_r+1} + \sum_{r\in R_2} (ly_r - y_r)\frac{W_r(x_r+1)}{x_r+1},\\
&\geq \sum_{r\in R_2} (W_r(y_r) - W_r(x_r)) + (l-1)\sum_{r\in R_2} y_r \frac{W_r(x_r+1)}{x_r+1},\\
&\geq \sum_{r\in R_2} (W_r(y_r) - W_r(x_r)) + (l-1)\sum_{r\in R_2} y_r \frac{W_r(y_r)}{y_r},\\
&= l\sum_{r\in R_2} W_r(y_r) - \sum_{r\in R_2} W_r(x_r),
\end{aligned}$$

where the final inequality follows from the discrete concavity of W_r and the fact that $y_r \geq x_r+1$ for every $r \in R_2$. Plugging into (18) and simplifying gives us

$$W(a^{\rm ne}; ln) \quad \geq \quad \sum_{r\in R_1} g_r \frac{W_r(x_r)}{x_r} + l\sum_{r\in R_2} W_r(y_r). \quad (19)$$

By the statement of the theorem, we are done unless $W(a^{\rm opt}; n) \geq W(a^{\rm ne}; ln)$, so suppose that this is the case. For each resource $r \in R_1$, multiply the welfare function W_r by a constant λ_r which satisfies

$$\lambda_r \frac{W_r(x_r)}{x_r} = \min_{r\in R_2} \frac{W_r(x_r+1)}{x_r+1}.$$

Since the allocation $\{x_1, \dots, x_m\}$ represents an allocation for a pure Nash equilibrium, we know that $\lambda_r \leq 1$ for all $r \in R_2$. Since $x_r \geq y_r$ for all resource $r \in R_1$, this welfare function scaling decreases the welfare of $a^{\rm ne}$ by at least as much as $a^{\rm opt}$. Since $W(a^{\rm opt}; n) \geq W(a^{\rm ne}; ln)$, this decrease only increases the ratio $W(a^{\rm opt}; n)/W(a^{\rm ne}; ln)$.

The point is that we can assume, for the remainder of the analysis, that all resources in R_1 have a common value of $W_r(x_r)/x_r$. Recalling that $\sum_{r\in R_1} g_r = l\sum_{r\in R_1} y_r$ and continuing the above derivation yields

$$\begin{aligned}
W(a^{\rm ne}; ln) \quad &\geq \quad \sum_{r\in R_1} g_r \frac{W_r(x_r)}{x_r} + l\sum_{r\in R_2} W_r(y_r),\\
&= \quad \sum_{r\in R_1} ly_r \frac{W_r(x_r)}{x_r} + l\sum_{r\in R_2} W_r(y_r),\\
&= \quad \sum_{r\in R_1} ly_r \frac{W_r(x_r)}{x_r} + lW(a^{\rm opt}; n) - l\sum_{r\in R_1} W_r(y_r),\\
&= \quad l\left[W(a^{\rm opt}; n) - \sum_{r\in R_1} W_r(x_r)\left(\frac{W_r(y_r)}{W_r(x_r)} - \frac{y_r}{x_r}\right)\right],\\
&\geq \quad l\left(W(a^{\rm opt}; n) - \gamma(ln)W(a^{\rm ne}; ln)\right).
\end{aligned}$$

Rearranging terms proves that

$$\frac{W(a^{\rm opt}; n)}{W(a^{\rm ne}; ln)} \leq \max\left\{1, \frac{1}{l} + \gamma(ln)\right\}.$$

■

With regards to the bound set forth in the previous theorem, it seems natural that the bound of $\frac{1}{l} + \gamma(ln)$ may hold in general irrespective of whether $W(a^{\rm opt}; n) \geq W(a^{\rm ne}; ln)$. The following example demonstrates that such intuition is false.

Example 3 *Consider a situation with two resource r_1 and r_2 with resource specific welfare functions $\sqrt{x}$ and $\sqrt{4x/5}$ respectively. Let $n = 2$ and $l = 4$. Then $1/l + \gamma(ln) = 0.5$ as $\gamma(ln) = 0.25$. One can observe this by noticing that $\sqrt{1}/\sqrt{4} - 1/4 = 0.25$. In this example, $a^{\rm opt}$, which consists of 2 players, has one player on each resource for a total welfare of $1 + \sqrt{4/5}$. The Nash allocation, $a^{\rm ne}$, which consists of 8 players, has 5 players on resource r_1 and 3 players on resource r_2 for a total welfare of $\sqrt{5} + \sqrt{12/5}$. Note that $W(a^{\rm opt}; 2)/W(a^{\rm ne}; 8) \approx 0.5005$ which violates the above bound.*

III. The Asymmetric Case

In this section we focus on deriving more general efficiency bounds for resource allocation problems with asymmetric agents. By asymmetric, we mean that each agent's action set is not necessarily identical, i.e., $\mathcal{A}_i$ need not equal $\mathcal{A}_j$ for all agents $i, j \in N$. Here, we focus on characterizing efficiency bounds for a more general class of equilibria, termed *coarse correlated equilibria*, which pertain to distributions over the joint action set $\mathcal{A}$. More specifically, consider a joint distribution $z = \{z_a\}_{a\in\mathcal{A}} \in \Delta(\mathcal{A})$ where $1 \geq z_a \geq 0$ denotes the component of the distribution associated with the joint action profile a. A joint distribution $z \in \Delta(\mathcal{A})$ is a coarse correlated equilibria if for all agents $i \in N$ and actions $a_i' \in \mathcal{A}_i$ [21]:

$$\sum_{a\in\mathcal{A}} U_i(a) z_a \geq \sum_{a\in\mathcal{A}} U_i(a_i', a_{-i}) z_a. \tag{20}$$

The set of coarse correlated equilibria, which are commonly referred to as no-regret points, contain the set of correlated equilibria, mixed Nash equilibria, as well pure Nash equilibria. The question that we focus on in this section is how the efficiency bounds derived in Section II extend to this broader class of equilibria where the welfare associated with a joint distribution $z \in \Delta(\mathcal{A})$ is taken as the expected welfare, i.e.,

$$W(z) = \sum_{a\in\mathcal{A}} W(a) z_a.$$

The importance of these bounds stem from the existence of simple and efficient dynamical processes which converge to the set of coarse correlated equilibria for any game [37]–[39].

A. A Motivating Example

We begin by focusing on whether the efficiency bounds for pure Nash equilibria given in Theorems 2 and 3 hold for the more general setting considered here. The following two examples shows that relaxations in either direction, i.e., considering pure Nash equilibria with asymmetric agents or considering mixed Nash equilibria with symmetric agents, violates the derived bounds.

Example 4 (Asymmetric Action Sets) *Consider a situation with $n+1$ resources and n asymmetric players where each player i can select either resource i or $i+1$. The resource specific welfare functions are of the form $W_{r_1}(x) = \sqrt{x}$ and $W_{r_k}(x) = \sqrt{x} \cdot (1/\sqrt{2})^{k-2}$ for all $k \geq 2$ and $x \geq 1$. The price of anarchy bound for pure Nash equilibria and symmetric agents given in Theorem 2 is $5/4$. For the asymmetric case, there exists a pure Nash equilibrium where each agent i selects resource $i+1$ which yields a total welfare is $1/(1-1/\sqrt{2})$ as $n \to \infty$. The optimal allocation is when each agent i selects resource i which yields a total welfare of $(2-1/\sqrt{2})/(1-1/\sqrt{2})$ as $n \to \infty$. This lower bounds the price of anarchy as $2 - 1/\sqrt{2} \approx 1.29$ which violates the price of anarchy bound for symmetric agents given in Theorem 2.*

Example 5 (Mixed Nash equilibria) *Consider a situation with 3 symmetric players, 3 resources, and each resource has an identical welfare function of the form $W_r(x) = x^{0.9}$. The optimal welfare associated with this setting is 3 achieved when each agent selects a distinct resource. The price of anarchy bound for pure Nash equilibria and symmetric agents given in Theorem 2 is $1.0387 = 1 + (1)/(3^{0.9}) - 1/3$. There is a mixed Nash equilibrium where each agent selects the resources with a uniform strategy $(1/3, 1/3, 1/3)$. This mixed Nash equilibrium yields an expected welfare of 2.876 and an efficiency ratio of $3/2.876 = 1.0431$. Hence, this efficiency bound violates the price of anarchy bound for symmetric agents given in Theorem 2.*

B. Preliminaries: Smoothness Arguments

"Smoothness arguments" represent an effective methodology for proving price of anarchy bounds for more general equilibrium concepts than that of pure Nash equilibria [40]. A smoothness argument requires proving that for some constants $\lambda,\ \mu > 0$ the following holds: for every pair of allocations $a, a^* \in \mathcal{A}$ (equilibrium, optimal, and otherwise),

$$\sum_{i=1}^{n} U_i(a_i^*, a_{-i}) \geq \lambda \cdot W(a^*) - \mu \cdot W(a). \tag{21}$$

By [40], establishing such a $\lambda,\ \mu > 0$ implies a price of anarchy bound of $(1+\mu)/\lambda$ for all mixed Nash equilibria, correlated equilibria, and coarse correlated equilibria. We will refer to such a price of anarchy bound as the *robust price of anarchy*. It is important to note that the derivations given in Section II do not utilize smoothness arguments as only the optimal allocation a^{opt} and Nash allocation a^{ne} were considered. Furthermore, such smoothness arguments also hold for establishing bicriteria bounds where there are variations in the number of agents in the two allocations a and a^*.

C. Bounding the efficiency of coarse correlated equilibria

We begin by proving that the price of anarchy bound of 2 in Theorem 1 also covers all mixed Nash equilibria, correlated equilibria, and coarse correlated equilibria.

Theorem 4 *Consider any single selection resource allocation game with n agents, asymmetric action sets, the Shapley value utility design, and submodular welfare functions. The robust price of anarchy is bounded above by 2.*

Proof: Let a and a' represent any two allocations and let $|a| = \{x_1, ..., x_m\}$ and $|a'| = \{y_1, ..., y_m\}$. Let z_r denote the number of agents that select resource r in both a and a', i.e.,

$$z_r = |\{i \in N : a_i = r\} \cap \{i \in N : a_i' = r\}|.$$

Note that $z_r \leq \min\{x_r, y_r\}$ for all resources $r \in R$. The proof of Theorem 2 demonstrates that

$$\begin{aligned}\sum_{i\in N} U_i(a_i', a_{-i}) & \qquad\qquad (22)\\ \geq W(a') + \sum_{r\in R_1} &\left(z_r \frac{W_r(x_r)}{x_r} + (y_r - z_r)\frac{W_r(x_r+1)}{x_r+1} - W_r(y_r) \right) \\ + \sum_{r\in R_2} &\left(z_r \frac{W_r(x_r)}{x_r} + (x_r - z_r)\frac{W_r(x_r+1)}{x_r+1} - W_r(x_r) \right)\end{aligned}$$

where R_1 and R_2 denote the resources on which $x_r \geq y_r$ and $x_r < y_r$, respectively. Using the fact that (22) is increasing in z_r, setting $z_r = 0$ for each $r \in R$ gives us the lower bound

$$\begin{aligned}
&\sum_{i \in N} U_i(a_i', a_{-i}) \\
&\quad \geq W(a') + \sum_{r \in R_1} \left(y_r \frac{W_r(x_r+1)}{x_r+1} - W_r(y_r) \right) \\
&\qquad + \sum_{r \in R_2} \left(x_r \frac{W_r(x_r+1)}{x_r+1} - W_r(x_r) \right), \qquad (23) \\
&\quad \geq W(a') + \sum_{r \in R_1} y_r \frac{W_r(x_r+1)}{x_r+1} - \sum_{r \in R_1} W_r(y_r) - \sum_{r \in R_2} W_r(x_r), \\
&\quad \geq W(a') + \sum_{r \in R_1} \left(W_r(y_r + x_r) - W_r(x_r) \right) \\
&\qquad - \sum_{r \in R_1} W_r(y_r) - \sum_{r \in R_2} W_r(x_r), \qquad (24) \\
&\quad \geq W(a') - W(a). \qquad (25)
\end{aligned}$$

where (24) stems from the discrete concavity of W_r and (25) stems from the fact that W_r is nondecreasing. Accordingly, the game is smooth with parameter $\lambda = 1$ and $\mu = 1$. Therefore, the robust price of anarchy is 2. ■

Theorem 4 proves that the efficiency bound of 2 holds for all coarse correlated equilibria even in the setting with asymmetric players. The following theorem demonstrates that we can provide a tighter characterization of the robust price of anarchy by exploiting the structure of the objective functions.

Theorem 5 *Consider any single selection resource allocation game with n agents, asymmetric action sets, the Shapley value utility design, and submodular welfare functions. The robust price of anarchy is bounded above by*

$$1 + \max_{r \in R} \max_{k \leq m \leq n} \max \left\{ \frac{W_r(k)}{W_r(m)} - \beta(n) \frac{k}{m}, 1 - \beta(n) \right\} \qquad (26)$$

where

$$\beta(n) = \min_{r \in R} \min_{1 \leq x \leq n} \frac{W_r(x+1)/(x+1)}{W_r(x)/x}.$$

Proof: Consider the proof of Theorem 4 up to equation (23). The first nontrivial term in (23) can be bounded below by

$$\begin{aligned}
&\sum_{r \in R_1} \left(y_r \frac{W_r(x_r+1)}{x_r+1} - W_r(y_r) \right) \\
&= \quad -\sum_{r \in R_1} W_r(x_r) \left[\frac{W_r(y_r)}{W_r(x_r)} - \frac{y_r}{x_r} \left(\frac{W_r(x_r+1)}{W_r(x_r)} \frac{x_r}{x_r+1} \right) \right], \\
&\geq \quad -\sum_{r \in R_1} W_r(x_r) \left[\frac{W_r(y_r)}{W_r(x_r)} - \frac{y_r}{x_r} \beta(n) \right], \\
&\geq \quad -\sum_{r \in R_1} W_r(x_r) \cdot \max_{r \in R} \max_{k \leq m \leq n} \left(\frac{W_r(k)}{W_r(m)} - \beta(n) \frac{k}{m} \right), \\
&\geq \quad -W(a) \cdot \max_{r \in R} \max_{k \leq m \leq n} \left(\frac{W_r(k)}{W_r(m)} - \beta(n) \frac{k}{m} \right).
\end{aligned}$$

The second nontrivial term in (23) can be bounded below by

$$\begin{aligned}
&\sum_{r \in R_2} \left(x_r \frac{W_r(x_r+1)}{x_r+1} - W_r(x_r) \right) \\
&\qquad = -\sum_{r \in R_2} W_r(x_r) \left[1 - \left(\frac{W_r(x_r+1)}{W_r(x_r)} \frac{x_r}{x_r+1} \right) \right], \\
&\qquad \geq -\sum_{r \in R_2} W_r(x_r) \left(1 - \beta(n) \right), \\
&\qquad \geq -W(a) \cdot (1 - \beta(n)).
\end{aligned}$$

Accordingly, we have that for any action profiles $a, a' \in \mathcal{A}$,

$$\sum_{i \in N} U_i(a_i', a_{-i}) \geq W(a') - \delta(n) W(a)$$

where

$$\delta(n) = \max_{r \in R} \max_{k \leq m \leq n} \max \left\{ \frac{W_r(k)}{W_r(m)} - \beta(n) \frac{k}{m}, 1 - \beta(n) \right\}.$$

Hence, the robust price of anarchy is $1 + \delta(n)$. ■

First, it is important to highlight that $\delta(n) \geq \gamma(n)$ as expected since the price of anarchy associated with pure Nash equilibria cannot be larger than the price of anarchy associated with a broader class of equilibria. For example, in the case of constant welfare functions, i.e., $W_r(x) = c$ for all $x \geq 1$, we have that

$$\begin{aligned}
\gamma(n) &= \max_{k \leq m \leq n} \left(\frac{W_r(k)}{W_r(m)} - \frac{k}{m} \right) \\
&= 1 - \frac{1}{n}
\end{aligned}$$

and

$$\begin{aligned}
\delta(n) &= \max_{k \leq m \leq n} \max \left\{ \frac{W_r(k)}{W_r(m)} - \beta(n) \frac{k}{m}, 1 - \beta(n) \right\} \\
&= \max \left\{ 1 - \frac{1}{2n}, \frac{1}{2} \right\} \\
&= 1 - \frac{1}{2n}.
\end{aligned}$$

Here, $\delta(n)$ uses the fact that $\beta(n) = 1/2$ where the minimizer is $x = 1$.

The following theorem shows that the robust price of anarchy bound presented in Theorem 5 can be further improved by exploiting the following property in the above derivation. Consider any two allocations a and a' and suppose there are x players using resource r in a and y players using resource r in a'. If $x + y > n$, then the number of players using both resource r in both a and a' is at least $x + y - n$. The following theorem makes this idea precise.

Theorem 6 *Consider any single selection resource allocation game with n agents, asymmetric action sets, the Shapley value utility design, and submodular welfare functions. The robust price of anarchy is bounded above by $1 + \eta(n)$ where*

$$\eta(n) = \max \left\{ \begin{array}{l} \max\limits_{r \in R, k \leq m \leq n} \left(\frac{W_r(k)}{W_r(m)} - \left(\frac{\max\{m+k-n, 0\} + \min\{n-m, k\} \cdot \tilde{\beta}_r(m)}{m} \right) \right), \\ \max\limits_{r \in R, k \leq m \leq n} \left(1 - \left(\frac{\max\{k+m-n, 0\} + \min\{n-m, k\} \cdot \tilde{\beta}_r(k)}{k} \right) \right) \end{array} \right.$$

where

$$\tilde{\beta}_r(m) = \frac{m}{m+1}\frac{W_r(m+1)}{W_r(m)}.$$

Proof: Consider the proof of Theorem 4 up to equation (22). One way to slightly optimize the analysis in the proof of Theorem 5 is to observe that if $x_r + y_r > n$, then the number z_r of shared players is at least $x_r + y_r - n$. Accordingly, the first nontrivial term in (22) can be bounded below by

$$\begin{aligned}
&\sum_{r\in R_1} z_r \frac{W_r(x_r)}{x_r} + (y_r - z_r)\frac{W_r(x_r+1)}{x_r+1} - W_r(y_r) \\
&= \sum_{r\in R_1} z_r\left(\frac{W_r(x_r)}{x_r} - \frac{W_r(x_r+1)}{x_r+1}\right) + y_r\frac{W_r(x_r+1)}{x_r+1} - W_r(y_r), \\
&\geq \sum_{r\in R_1}\Bigg(\max\{x_r+y_r-n,0\}\left(\frac{W_r(x_r)}{x_r} - \frac{W_r(x_r+1)}{x_r+1}\right) \\
&\qquad + y_r\frac{W_r(x_r+1)}{x_r+1} - W_r(y_r)\Bigg), \\
&= \sum_{r\in R_1}\Bigg(\max\{x_r+y_r-n,0\}\frac{W_r(x_r)}{x_r} \\
&\qquad + \min\{n-x_r,y_r\}\frac{W_r(x_r+1)}{x_r+1} - W_r(y_r)\Bigg), \\
&= -\sum_{r\in R_1} W_r(x_r)\Bigg(\frac{W_r(y_r)}{W_r(x_r)} - \frac{\max\{x_r+y_r-n,0\}}{x_r} \\
&\qquad - \frac{\min\{n-x_r,y_r\}}{x_r+1}\frac{W_r(x_r+1)}{W_r(x_r)}\Bigg) \\
&\geq -W(a)\cdot\max_{r\in R}\max_{k\leq m\leq n}\Bigg(\frac{W_r(k)}{W_r(m)} - \frac{\max\{m+k-n,0\}}{m} \\
&\qquad - \frac{\min\{n-m,k\}\cdot\tilde{\beta}_r(m)}{m}\Bigg).
\end{aligned}$$

where the first inequality stems from the fact that $W_r(x_r)/x_r \geq W_r(x_r+1)/(x_r+1)$ and the last inequality stems from the fact that $x_r \geq y_r$ for all resources $r \in R_1$. The second nontrivial term in (22) can be bounded below by

$$\begin{aligned}
&\sum_{r\in R_2}\left(z_r\frac{W_r(x_r)}{x_r} + (x_r - z_r)\frac{W_r(x_r+1)}{x_r+1} - W_r(x_r)\right) \\
&\geq \sum_{r\in R_2}\Bigg(\max\{x_r+y_r-n,0\}\frac{W_r(x_r)}{x_r} \\
&\qquad + \min\{n-y_r,x_r\}\frac{W_r(x_r+1)}{x_r+1} - W_r(x_r)\Bigg), \\
&= -\sum_{r\in R_2} W_r(x_r)\Bigg(1 - \frac{\max\{x_r+y_r-n,0\}}{x_r} \\
&\qquad - \frac{\min\{n-y_r,x_r\}}{x_r+1}\frac{W_r(x_r+1)}{W_r(x_r)}\Bigg), \\
&= -\sum_{r\in R_2} W_r(x_r)\Bigg(1 - \frac{\max\{x_r+y_r-n,0\}}{x_r} \\
&\qquad - \frac{\min\{n-y_r,x_r\}\tilde{\beta}_r(x_r)}{x_r}\Bigg), \\
&\geq -W(a)\cdot\max_{r\in R}\max_{k\leq m\leq n}\Bigg(1 - \frac{\max\{k+m-n,0\}}{k} \\
&\qquad - \frac{\min\{n-m,k\}\cdot\tilde{\beta}_r(k)}{k}\Bigg),
\end{aligned}$$

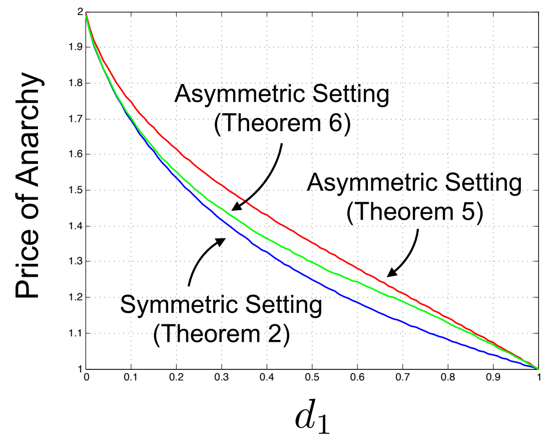

Fig. 3. Consider the resource allocation problem depicted in Example 1 where each resource $r_j \in R$ has a submodular objective of the form $W_{r_j}(x) = x^{d_j}$ where $d_j \in [0,1]$ and each agent $i \in N$ is assigned a utility function in accordance with the Shapley value. Without loss of generality let $0 \leq d_1 \leq \ldots \leq d_n \leq 1$. This figure highlights the robust price of anarchy for both the symmetric and asymmetric settings as a function of d_1 where d_1 varies between $[0,1]$ and when there are 100 agents. Theorem 2 provides the price of anarchy bound associated with pure Nash equilibria in the symmetric setting while Theorems 5 and 6 provide the price of anarchy bound associated with coarse correlated equilibria in the asymmetric setting. Note that the gap between the two bounds is relatively small. However, there is definitely a gap as highlighted by Examples 4 and 5.

where the last inequality stems from the fact that $x_r < y_r$ for all resources $r \in R_2$. Therefore, we have that

$$\sum_{i\in N} U_i(a_i', a_{-i}) \leq W(a) - \eta(n)W(a')$$

which completes the proof. ■

Figure 3 illustrates the bounds derived in Theorems 5 and 6 bounds on the resource allocation problem given in Example 1.

D. A Bicriteria Bound for Coarse Correlated Equilibria

In this section we derive bicriteria bounds for resource allocation problems with asymmetric players. Unlike the analysis presented in Section II-C, the forthcoming bounds will hold for a broad class of equilibria including coarse correlated equilibria. Hence, we will refer to this bound as the robust bicriteria bound. As before, we will consider situations where there are n players in the optimal allocation and ln players in the equilibrium allocation where $l \in \{1, 2, \ldots\}$. Since players are asymmetric in this setting, we will focus on the case where each player in the optimal allocation has l "copies" in the equilibrium allocation.

Theorem 7 *Consider any anonymous single selection resource allocation game with asymmetric action sets, the Shalpey value utility design, and submodular welfare functions. The robust bicriteria bound associated with any coarse correlated equilibrium with ln players and an optimal allocation with n players where $l \in \{1, 2, \ldots\}$ is bounded above by $1 + \frac{1}{l}$.*

Proof: Let a be an action profile for the game consisting of ln players and a' be an action profile for the game consisting of n players. For notational simplicity, define $|a| = \{x_1, ..., x_m\}$ and $|a'| = \{y_1, ..., y_m\}$. Here, we have $\sum_{k=1}^m x_k = ln$ and $\sum_{k=1}^m y_k = n$. Assume that for each $j \in \{0, \dots, l-1\}$, the players $i \in \{jn+1, \dots, (j+1)n\}$ are in bijective correspondence with the players in the original game. We write "a_i'" even when $i > n$ with the understanding that i means $[(i-1) \bmod n] + 1$. Accordingly, we have

$$\begin{aligned}\sum_{j=0}^{l-1}\left(\sum_{i=jn+1}^{(j+1)n} U_i(a_i', a_{-i})\right) &\geq \sum_{j=0}^{l-1}\left(\sum_{r\in R} y_r \frac{W_r(x_r+1)}{x_r+1}\right),\\ &\geq \sum_{j=0}^{l-1}\left(\sum_{r\in R} W_r(x_r+y_r) - W_r(x_r)\right),\\ &\geq l\cdot W(a') - l\cdot W(a).\end{aligned}$$

where the second inequality follows from the discrete concavity of W_r and the third inequality follows from the fact that W_r is increasing. Hence, the game is smooth with parameters $\lambda = l$ and $\mu = l$ which gives a robust bicriteria bound of $1 + 1/l$. ∎

The following theorem strengthens the bound presented in Theorem 7 by exploiting the structure of the objective functions.

Theorem 8 *Consider any anonymous single selection resource allocation game with asymmetric action sets, the Shalpey value utility design, and submodular welfare functions. The robust bicriteria bound associated with any coarse correlated equilibrium with ln players and an optimal allocation with n players where $l \in \{1, 2, \dots\}$ is bounded above by $1/l + \eta(l, n)$ where $\eta(l, n)$ is defined as*

$$\max\begin{cases}\displaystyle\max_{r\in R, k\leq \min\{m_1,n\}, m_1\leq ln}\left(\frac{W_r(k)}{W_r(m_1)} - \frac{k}{m_1+1}\frac{W_r(m_1+1)}{W_r(m_1)}\right),\\ \displaystyle\max_{r\in R,\ m_2\leq ln}\left(1 - \frac{m_2}{m_2+1}\frac{W_r(m_2+1)}{W_r(m_2)}\right).\end{cases} \tag{28}$$

Proof: Consider the setup for the proof of Theorem 7. As before, define $R_1 = \{r \in R : x_r \geq y_r\}$ and $R_2 = \{r \in R : x_r < y_r\}$. Incorporating R_1 and R_2 into the above proof gives us for any $j \in \{0, \dots, l-1\}$

$$\begin{aligned}\sum_{i=jn+1}^{(j+1)n} U_i(a_i', a_{-i}) &\geq \sum_{r\in R} y_r \frac{W_r(x_r+1)}{x_r+1},\\ &= \sum_{r\in R_1} y_r \frac{W_r(x_r+1)}{x_r+1} + \sum_{r\in R_2} x_r \frac{W_r(x_r+1)}{x_r+1}\\ &\qquad + \sum_{r\in R_2}(y_r - x_r)\frac{W_r(x_r+1)}{x_r+1}, \qquad (29)\\ &\geq \sum_{r\in R_1} y_r \frac{W_r(x_r+1)}{x_r+1} + \sum_{r\in R_2} x_r \frac{W_r(x_r+1)}{x_r+1}\\ &\qquad + \sum_{r\in R_2}(W_r(y_r) - W_r(x_r)), \qquad (30)\\ &= W(a') - \sum_{r\in R_1}\left(W_r(y_r) - y_r\frac{W_r(x_r+1)}{x_r+1}\right)\\ &\qquad - \sum_{r\in R_2}\left(W_r(x_r) - x_r\frac{W_r(x_r+1)}{x_r+1}\right). \qquad (31)\end{aligned}$$

Focusing on the first non-trivial term in (31) we have

$$\begin{aligned}&\sum_{r\in R_1}\left(W_r(y_r) - y_r\frac{W_r(x_r+1)}{x_r+1}\right)\\ &= \sum_{r\in R_1} W_r(x_r)\left(\frac{W_r(y_r)}{W_r(x_r)} - \frac{y_r}{x_r+1}\frac{W_r(x_r+1)}{W_r(x_r)}\right),\\ &\leq W(a)\max_{r\in R,\ k\leq\min\{m_1,n\}, m_1\leq ln}\left(\frac{W_r(k)}{W_r(m_1)} - \frac{k}{m_1+1}\frac{W_r(m_1+1)}{W_r(m_1)}\right).\end{aligned}$$

Focusing on the second non-trivial term in (31) we have

$$\begin{aligned}&\sum_{r\in R_2}\left(W_r(x_r) - x_r\frac{W_r(x_r+1)}{x_r+1}\right)\\ &= \sum_{r\in R_2} W_r(x_r)\left(1 - \frac{W_r(x_r+1)}{W_r(x_r)}\frac{x_r}{x_r+1}\right),\\ &\leq W(a)\max_{r\in R,\ m_2\leq ln}\left(1 - \frac{m_2}{m_2+1}\frac{W_r(m_2+1)}{W_r(m_2)}\right).\end{aligned}$$

Therefore, from (31) we obtain

$$\begin{aligned}\sum_{j=0}^{l-1}\left(\sum_{i=jn+1}^{jn} U_i(a_i', a_{-i})\right) &\geq \sum_{j=0}^{l-1}\left(W(a') - \eta(l,n)\cdot W(a)\right),\\ &= l\cdot W(a') - l\cdot\eta(l,n)\cdot W(a).\end{aligned}$$

Hence, the game is smooth with parameters $\lambda = l$ and $\mu = l\cdot\eta(l, n)$ which gives a robust bicriteria bound of $1/l + \eta(l, n)$. ∎

Note that this bound is below 1 for welfare functions which η is sufficiently small. For example, for linear welfare functions, $\eta(l, n) = 0$ for all l and n. Hence, we recover the tight bound of $1/l$. See Figure 4 for an illustration of the bounds derived in this Section.

IV. Illustrating Example: The Vehicle Target Assignment Problem

In this section we apply the theoretical developments in this paper to a special class of the vehicle target assignment problem introduced in Section I-A. The two variations from the setup provided in Section I-A are as follows: (i) each vehicle $i \in N$ has a common detection/destroy probability $p \in [0, 1]$ and (ii) each vehicle can assign itself to just a single target, i.e., $\mathcal{A}_i \subseteq R$. To study the impact of the results contained within this paper, we analyze a game-theoretic formulation of the vehicle target assignment where each vehicle is assigned a Shapley value utility of the form

$$U_i(a_i = r, a_{-i}) = \frac{v_r\left(1 - (1-p)^{|a|_r}\right)}{|a|_r}. \tag{32}$$

Such a design approach will guarantee the existence of a pure Nash equilibrium irrespective of the number of vehicles, the structure of the action sets, or the number or relative worth of the targets. The value of this work is that it provides a *systematic* approach for evaluating the efficiency of the resulting equilibria for a broad class of resource allocation

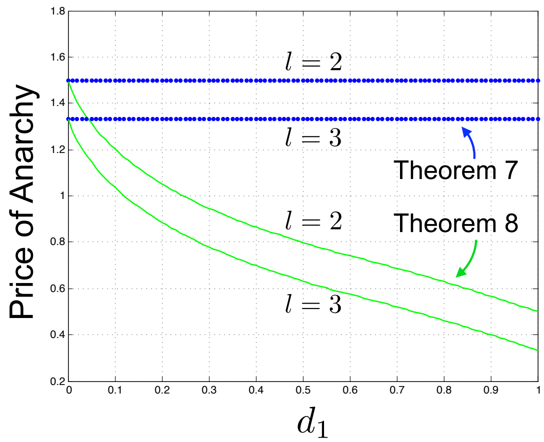

Fig. 4. Consider the resource allocation problem depicted in Example 1 where each resource $r_j \in R$ has a submodular objective of the form $W_{r_j}(x) = x^{d_j}$ where $d_j \in [0, 1]$ and each agent $i \in N$ is assigned a utility function in accordance with the Shapley value. Without loss of generality let $0 \leq d_1 \leq \ldots \leq d_n \leq 1$. This figure highlights the relative price of anarchy for the asymmetric settings as a function of d_1 where d_1 varies between $[0, 1]$ when there are 100 agents for $l = 2$ and $l = 3$. The blue dotted lines indicate the the bounds provided in Theorem 7 while the green lines indicate the optimized bounds derived in Theorem 8.

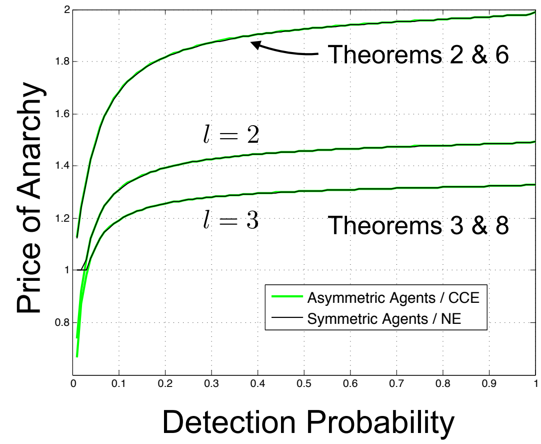

Fig. 5. This figure presents the price of anarchy as a function of detection probability for the vehicle target assignment problem with 100 vehicles and the Shapley value utility design. For the situation consisting of an equal number of vehicles in both the equilibrium and optimal allocation, it turns out that the efficiency bounds provided by Theorems 2 and 6 are identical as highlighted above. For the bicriteria bounds, we plotted the case when $l = 2$ and $l = 3$. For these situations, the efficiency bounds provided by Theorems 3 and 8 are close but not identical. As expected, the efficiency bound provided by Theorem 3 is less than the efficiency bound provided by Theorem 8 for all situations which resulted in a price of anarchy ≥ 1.

problems including the presented vehicle target assignment problem. The characterization of the price of anarchy is provided in the following corollary.

Corollary 9 *Consider any single selection vehicle target assignment problem where each vehicle $i \in N$ has a common detection probability p and a utility function in accordance with the vehicle's Shapley value as in (32). If the vehicles are symmetric, then the price of anarchy for pure Nash equilibria is bounded above by*

$$1 + \max_{k \leq m \leq n} \left(\frac{1-(1-p)^k}{1-(1-p)^m} - \frac{k}{m} \right) \tag{33}$$

and the bicriteria bound for any $l \in \{1, 2, \ldots\}$ is bounded above by

$$\max \left\{ 1, \frac{1}{l} + \max_{k \leq m \leq ln} \left(\frac{1-(1-p)^k}{1-(1-p)^m} - \frac{k}{m} \right) \right\}. \tag{34}$$

If the vehicles are asymmetric, then the robust price of anarchy is bounded above by

$$1+\max \left\{ \begin{array}{l} \max\limits_{k \leq m \leq n} \left(\frac{1-(1-p)^k}{1-(1-p)^m} - \frac{\max\{m+k-n,0\}}{m} - \frac{\min\{n-m,k\} \cdot \tilde{\beta}_r(m)}{m} \right) \\ \max\limits_{k \leq m \leq n} \left(1 - \left(\frac{\max\{k+m-n,0\}+\min\{n-m,k\} \cdot \tilde{\beta}_r(k)}{k} \right) \right), \end{array} \right. \tag{35}$$

and the robust bicriteria bound for any $l \in \{1, 2, \ldots\}$ is bounded above by

$$1/l+\max \left\{ \begin{array}{l} \max\limits_{k \leq \min\{m_1,n\}, m_1 \leq ln} \left(\frac{1-(1-p)^k}{1-(1-p)^{m_1}} - \frac{k}{m_1+1} \frac{1-(1-p)^{m_1+1}}{1-(1-p)^{m_1}} \right) \\ \max\limits_{k \leq m_2 \leq ln} \left(1 - \frac{m_2}{m_2+1} \frac{1-(1-p)^{m_2+1}}{1-(1-p)^{m_2}} \right), \end{array} \right. \tag{36}$$

where

$$\tilde{\beta}_r(m) = \left(\frac{m}{m+1} \right) \left(\frac{1-(1-p)^{m+1}}{1-(1-p)^m} \right).$$

In the above corollary, (33) follows from Theorem 2, (34) follows from Theorem 3, (35) follows from Theorem 6, and (36) follows from Theorem 8.

Figure 5 plots the above efficiency guarantees for the vehicle target assignment problem with 100 vehicles and a common detection probability p ranging from 0 to 1. Notice that the gap between the efficiency guarantees for pure Nash in the symmetric setting versus coarse correlated equilibria in the asymmetric setting is virtually non-existent. It turns out that this bound matches the price of anarchy bound for single selection anonymous vehicle target assignment problems as derived in [16]. However, the price of anarchy in [16] was derived explicitly for the specific vehicle target assignment problem and hence has limited ability to be extended beyond that domain. Using the results in this paper, deriving this price of anarchy boils down to a systematic procedure and requires no tweaking for the specific domain.

V. Concluding Remarks

There is a large body of literature focused on characterizing the inefficiency of Nash equilibria for a wide array of application domains [41]. However, from a control-theoretic perspective these results are unsatisfying since utility functions can be designed in engineering systems. Hence, developing utility design methodologies to optimize the price of anarchy is of fundamental importance. This paper explores one promising utility design methodology for accomplishing this task – the Shalpey value. The results in this paper provide guarantees on the efficiency of the resulting equilibria when utilizing this

Shapley value utility design for a class of resource allocation problems.

One interesting direction for future research is to understand the role of budget balance in utility design. The Shapley value utility is budget-balanced, meaning that $\sum_{i \in N} U_i(a) = W(a)$ in every assignment a. There is no obvious motivation for requiring budget-balance in utility design and yet, at least in the case of the Shapley value, this property is correlated with good efficiency guarantees. Determining whether or not there is a deeper connection between these two properties is an intriguing open question.

More generally, with the goal of identifying the methodology that optimizes the price of anarchy, it is important that the presented analysis be extended to alternative utility design methodologies, such as the marginal contribution utility and weighted Shapley value [16].

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