Selfish Routing with Atomic Players

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Introduction. One of the most successful applications of the price of anarchy—the worst-case ratio between the objective function values of noncooperative equilibria and optima—is to "selfish routing", a classical model of how independent network users route traffic in a congested network. However, almost all existing work on this topic (e.g., [2, 5, 7]) assumes a large population of very small network users, so that the actions of a single individual have negligible impact on the cost incurred by others. This assumption—in game theory terminology, that the game is *nonatomic*—is obviously not justifiable in all applications.

In this note, we prove that the known upper bounds on the price of anarchy for nonatomic selfish routing games carry over to atomic selfish routing games, provided network users are permitted to route traffic fractionally over many paths. Qualitatively, this fact makes these existing bounds more robust: they do not depend on the assumption of a large population of network users, just on the assumption that traffic can be routed fractionally.

While this "atomic splittable" model has been studied before in the transportation science [1, 3], networking [4], and theoretical computer science [7] communities, these are the first bounds on the price of anarchy in this model.[1] The only other similar result for this model that we are aware of is one of Roughgarden and Tardos [7], who showed that their "bicriteria bound" for nonatomic selfish routing games carries over to the atomic splittable case.

Finally, we note that on a naive level our result is highly intuitive: moving from a nonatomic model to an atomic one can be viewed as identifying groups of previously independent and noncooperative traffic into single strategic agents. Shouldn't the inefficiency of the equilibrium be decreasing in the degree of cooperation? If this were always the case, the main result of this note would follow immediately from previous work on nonatomic selfish routing games. However, counterexamples to this intuition are known [1]. This note can be interpreted as proving this intuition correct from a *worst-case* perspective, rather than on an instance-by-instance basis.

The Model. By an *(atomic splittable) instance*, we mean a triple (G, r, c), where G is a directed graph with (not necessarily distinct) source vertices $\{s_1, \ldots, s_k\}$ and sink vertices $\{t_1, \ldots, t_k\}$; r is a vector indexed by source-sink pairs, where player i must fractionally route r_i units of traffic from s_i to t_i; and c is a vector of cost functions, one for each edge of G. As is standard, we will assume that each function c_e is a nonnegative, nondecreasing, and continuous function. To ease the exposition, we will also assume that each cost function is differentiable and convex. These assumptions, which hold in all of the most popular applications of the model, can be weakened somewhat.

For an instance (G, r, c), a *feasible flow* f comprises k nonnegative vectors $f^1, \ldots, f^k$, where f^i is defined on the s_i-t_i paths $\mathcal{P}_i$ of G and satisfies $\sum_{P \in \mathcal{P}_i} f^i_P = r_i$. For a flow f, $f_e = \sum_{i=1}^k \sum_{P \in \mathcal{P}_i} f^i_P$ denotes the total flow on edge e. The cost $c_P(f)$ of a path P with respect to a flow f is the sum $\sum_{e \in P} c_e(f_e)$ of the costs of its edges. The cost $C_i(f)$ to player i is defined by $\sum_{P \in \mathcal{P}_i} c_P(f) f^i_P$. The cost $C(f)$ of a flow f is defined by $\sum_{i=1}^k C_i(f)$ or, equivalently, $\sum_{e \in E} c_e(f_e) f_e$.

A flow f is *at Nash equilibrium*, or is a *Nash flow*, if for each player i, f^i minimizes $C_i(f)$ when the other flows $\{f^j\}_{j \neq i}$ are held fixed. When $k = 1$, the Nash equilibria are precisely the optimal flows. As $k \to \infty$ and $r_i \to 0$ for all players i, we recover the more well-studied nonatomic selfish routing game.

Previous work shows that every instance admits at least one Nash flow [3, 4]. The *price of anarchy* $\rho(G, r, c)$ of an instance (G, r, c) is $\sup C(f)/C(\hat{f})$, where the supremum ranges over Nash flows f and feasible flows

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[1] Very recently, Awerbuch, Azar, and Epstein (personal communication, June 2004) proved nearly matching upper and lower bounds on the price of anarchy in atomic selfish routing games where each player must route its traffic on a single path. In particular, they proved that the upper bounds of this paper no longer hold when fractional routing is not allowed.

$\hat{f}$. The uniqueness of Nash flows in atomic splittable instances is not well understood [3, 4], and we are therefore allowing the possibility that an instance admits multiple Nash flows. Note that a bound on the price of anarchy applies, by definition, to *all* Nash flows of an instance. We denote by $\alpha_k(\mathcal{C})$ the largest price of anarchy occurring in an instance with k players and cost functions in the set $\mathcal{C}$. Similarly, $\alpha_\infty(\mathcal{C})$ denotes the largest-possible price of anarchy in a nonatomic instance with cost functions in $\mathcal{C}$. The value $\alpha_\infty(\mathcal{C})$ is known for many sets $\mathcal{C}$: for example, it is 4/3 if $\mathcal{C}$ is the set of affine or concave functions [2, 7], and is $\Theta(d/\log d)$ if $\mathcal{C}$ is the set of polynomials with nonnegative coefficients and degree at most d [5].

The structure inherent in a Nash flow that permits bounds on the price of anarchy is identified in the next proposition. In the statement of the proposition and throughout this note, if f is feasible for (G, r, c), then $c_e^i(f_e)$ denotes the expression $c_e(f_e) + f_e^i \cdot c_e'(f_e)$. Intuitively, c_e^i is the cost of edge e from player i's perspective; see [7] for further discussion of this intuition.

PROPOSITION 1. *If f is a Nash flow for (G, r, c), then f minimizes $\sum_{i=1}^k \sum_{e \in E} c_e^i(f_e)\hat{f}_e^i$ over all feasible flows $\hat{f}$ for (G, r, c).*

Proposition 1 follows fairly easily from the definition of a Nash flow and our assumption that cost functions are convex, and we omit a formal proof.

Our Results. We now prove our main result, that the price of anarchy in atomic splittable instances is no larger than that in nonatomic instances. Formally, we will show the following, where a set $\mathcal{C}$ of cost functions is *inhomogeneous* if it contains some cost function c satisfying $c(0) > 0$.

THEOREM 2. *For all $k \geq 1$ and inhomogeneous sets $\mathcal{C}$ of cost functions, $\alpha_k(\mathcal{C}) \leq \alpha_\infty(\mathcal{C})$.*

Theorem 2 is tight in the sense that $\lim_{k\to\infty} \alpha_k(\mathcal{C}) \geq \alpha_\infty(\mathcal{C})$; this holds because every nonatomic instance can effectively be "simulated" by an atomic one with sufficiently many players. Since $\alpha_1(\mathcal{C}) = 1$ for all $\mathcal{C}$, Theorem 2 is not tight when $k = 1$. The case of intermediate k is not yet understood; see [6] for more details.

The key idea in our proof of Theorem 2 is the definition of an intermediate expression that is easily compared to both α_k and α_∞, two values that seem difficult to compare directly, in part due the counterintuitive examples of [1]. This intermediate expression, which we will unimaginatively call β_k, is defined as follows.

DEFINITION 3. *For a cost function c, $\beta_k(c) \equiv \sup\{f_e c_e(f_e)/[\hat{f}_e c_e(\hat{f}_e) + \sum_{i=1}^k (f_e^i - \hat{f}_e^i)c_e^i(f_e)]\}$, where the supremum is taken over all non-negative real numbers $f_e, f_e^1, \ldots, f_e^k, \hat{f}_e, \hat{f}_e^1, \ldots, \hat{f}_e^k$ such that $\sum_i f_e^i = f_e$ and $\sum_i \hat{f}_e^i = \hat{f}_e$. For a set $\mathcal{C}$ of cost functions, $\beta_k(\mathcal{C}) \equiv \sup_{c\in\mathcal{C}} \beta_k(c)$.*

Definition 3 is motivated by the so-called anarchy value of [2, 5] but does not seem to admit a simple interpretation. Nevertheless, as we will see, it fits snugly into the proof framework of [2, 5].

Proof of Theorem 2: Fix $k \geq 1$ and a set $\mathcal{C}$ of cost functions. We next show that $\alpha_k(\mathcal{C}) \leq \beta_k(\mathcal{C})$. Our proof of this will be along the same lines as those in [2, 5], and will rely on Proposition 1. Let (G, r, c) be an instance with k players and cost functions in the set $\mathcal{C}$. Let f be a Nash flow and $\hat{f}$ an optimal flow for (G, r, c). For each edge e, the definition of $\beta_k(\mathcal{C})$ implies that $\hat{f}_e c_e(\hat{f}_e) \geq f_e c_e(f_e)/\beta_k(\mathcal{C}) + \sum_{i=1}^k (\hat{f}_e^i - f_e^i)c_e^i(f_e)$. Summing over all of the edges and applying Proposition 1 then yields $C(\hat{f}) = \sum_{e\in E} \hat{f}_e c_e(\hat{f}_e) \geq \frac{1}{\beta_k(\mathcal{C})} \sum_{e\in E} f_e c_e(f_e) + \sum_{e\in E} \sum_{i=1}^k (\hat{f}_e^i - f_e^i)c_e^i(f_e) \geq C(f)/\beta_k(\mathcal{C})$, showing that $\alpha_k(\mathcal{C}) \leq \beta_k(\mathcal{C})$.

To complete the proof, we must show that $\beta_k(\mathcal{C}) \leq \alpha_\infty(\mathcal{C})$. This amounts to exhibiting, for every possible setting of the parameters in Definition 3, a nonatomic instance with sufficiently large price of anarchy. Details of this argument are omitted due to space constraints and can be found in [6]. ■

While this note effectively settles the price of anarchy in atomic splittable instances, interesting questions remain about such instances. See [6] for details.

References

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