CS261: Exercise Set #4

For the week of January 25–29, 2016

Instructions:

(1) *Do not turn anything in.*

(2) The course staff is happy to discuss the solutions of these exercises with you in office hours or on Piazza.

(3) While these exercises are certainly not trivial, you should be able to complete them on your own (perhaps after consulting with the course staff or a friend for hints).

Exercise 16

In Lecture #7 we noted that the maximum flow problem translates quite directly into a linear program:

$$\max \sum_{e \in \delta^+(s)} f_e$$

subject to

$$\sum_{e \in \delta^-(v)} f_e - \sum_{e \in \delta^+(v)} f_e = 0 \qquad \text{for all } v \neq s, t$$
$$f_e \leq u_e \qquad \text{for all } e \in E$$
$$f_e \geq 0 \qquad \text{for all } e \in E.$$

(As usual, we are assuming that s has no incoming edges.) In Lecture #8 we considered the following alternative linear program, where $\mathcal{P}$ denotes the set of s-t paths of G:

$$\max \sum_{P \in \mathcal{P}} f_P$$

subject to

$$\sum_{P \in \mathcal{P} \,:\, e \in P} f_P \leq u_e \qquad \text{for all } e \in E$$
$$f_P \geq 0 \qquad \text{for all } P \in \mathcal{P}.$$

Prove that these two linear programs always have equal optimal objective function value.

Exercise 17

In the *multicommodity flow problem*, the input is a directed graph $G = (V, E)$ with k source vertices $s_1, \ldots, s_k$, k sink vertices $t_1, \ldots, t_k$, and a nonnegative capacity u_e for each edge $e \in E$. An s_i-t_i pair is called a *commodity.* A *multicommodity flow* if a set of k flows $\mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(k)}$ such that (i) for each $i = 1, 2, \ldots, k$, $\mathbf{f}^{(i)}$ is an s_i-t_i flow (in the usual max flow sense); and (ii) for every edge e, the total amount of flow (summing over all commodities) sent on e is at most the edge capacity u_e. The *value* of a multicommodity flow is the sum of the values (in the usual max flow sense) of the flows $\mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(k)}$.

Prove that the problem of finding a multicommodity flow of maximum-possible value reduces in polynomial time to solving a linear program.

Exercise 18

Consider a primal linear program (P) of the form

$$\max \mathbf{c}^T \mathbf{x}$$

subject to

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0}. \end{aligned}$$

The recipe from Lecture #8 gives the following dual linear program (D):

$$\min \mathbf{b}^T \mathbf{y}$$

subject to

$$\begin{aligned} \mathbf{A}^T \mathbf{y} &\geq \mathbf{c} \\ \mathbf{y} &\in \mathbb{R}. \end{aligned}$$

Prove weak duality for primal-dual pairs of this form: the (primal) objective function value of every feasible solution to (P) is bounded above by the (dual) objective function value of every feasible solution to (D).[1]

Exercise 19

Consider a primal linear program (P) of the form

$$\max \mathbf{c}^T \mathbf{x}$$

subject to

$$\begin{aligned} \mathbf{A}\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

and corresponding dual program (D)

$$\min \mathbf{b}^T \mathbf{y}$$

subject to

$$\begin{aligned} \mathbf{A}^T \mathbf{y} &\geq \mathbf{c} \\ \mathbf{y} &\geq \mathbf{0}. \end{aligned}$$

Suppose $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are feasible for (P) and (D), respectively. Prove that if $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ do not satisfy the complementary slackness conditions, then $\mathbf{c}^T \hat{\mathbf{x}} \neq \mathbf{b}^T \hat{\mathbf{y}}$.

Exercise 20

Recall the linear programming relaxation of the minimum-cost bipartite matching problem:

$$\min \sum_{e \in E} c_e x_e$$

[1] In Lecture #8, we only proved weak duality for primal linear programs with only inequality constraints (and hence dual programs with nonnegative variables), like those in Exercise 19.

subject to

$$\sum_{e \in \delta(v)} x_e = 1 \qquad \text{for all } v \in V \cup W$$
$$x_e \geq 0 \qquad \text{for all } e \in E.$$

In Lecture #8 we appealed to the Hungarian algorithm to prove that this linear program is guaranteed to have an optimal solution that is 0-1. This point of this exercise is to give a direct proof of this fact, without recourse to the Hungarian algorithm.

(a) By a *fractional solution*, we mean a feasible solution to the above linear program such that $0 < x_e < 1$ for some edge $e \in E$. Prove that, for every fractional solution, there is an even cycle C of edges with $0 < x_e < 1$ for every $e \in C$.

(b) Prove that, for all ϵ sufficiently close to 0 (positive or negative), adding ϵ to x_e for every other edge of C and subtracting ϵ from x_e for the other edges of C yields another feasible solution to the linear program.

(c) Show how to transform a fractional solution $\mathbf{x}$ into another fractional solution $\mathbf{x}'$ such that: (i) $\mathbf{x}'$ has fewer fractional coordinates than $\mathbf{x}$; and (ii) the objective function value of $\mathbf{x}'$ is no larger than that of $\mathbf{x}$.

(d) Conclude that the linear programming relaxation above is guaranteed to possess an optimal solution that is 0-1 (i.e., not fractional).